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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

NINTH SESSION, 1890-91.

First Meeting, November 14th, 1890.

R. E. ALLARDICE, Esq., M.A., F.R.S.E., Vice-President, in the Chair.

For this Session the following Office-bearers were elected:—

President—Mr R. E. ALLARDICE, M.A., F.R.S.E.
Vice-President—Mr T. B. SPRAGUE, M.A., F.R.S.E.
Hon. Secretary—Mr JOHN ALISON, M.A., F.R.S.E.
Hon. Treasurer—Rev. JOHN WILSON, M.A., F.R.S.E.
Editor of Proceedings—Mr R. E. ALLARDICE.

Committee—

Professor STEGGALL, M.A., F.R.S.E.; Messrs JOHN WATT BUTTERS, JOHN B. CLARK, M.A.; ALEXANDER MORGAN, M.A., B.Sc.; J. T. MORRISON, M.A., B.Sc.; and WILLIAM WALLACE, M.A.

Some New Properties of the Triangle.

By J. S. Mackay, M.A., LL.D.

[The substance of this communication will be included in Dr Mackay's paper on *The Triangle* in the first volume of the *Proceedings*, which is about to be printed.]

Proofs of some optical theorems.

By William Peddie, D.Sc.

[The results of this paper will be contained in Dr Peddie's book on Physics, which will appear in a short time.]

Second Meeting, December 12th, 1890.

R. E. Allardice, Esq., President, in the Chair.

On the condition that the straight line $lx + my + nz = 0$ should be a normal to the conic $(a, b, c, f, g, h)(x, y, z)^2 = 0$ the co-ordinates being trilinear.

By R. H. Pinkerton, M.A.

1. The condition in question may be found by using the following theorem:—

If the equation in trilinear co-ordinates

$$F(x, y, z) \equiv (u, v, w, u', v', w')(x, y, z)^2 = 0 \quad \ldots \quad (A)$$

represents a pair of straight lines, then the line whose equation is

$$lx + my + nz = 0 \quad \ldots \quad \ldots \quad \ldots \quad (B)$$

will be perpendicular to one of those lines if

$$F(l - m\cos C - n\cos B,\ m - n\cos A - l\cos C,\ n - l\cos B - m\cos C) = 0 \quad (C)$$

where A, B, C are the angles of the fundamental triangle.

To prove this, transform the equations (A) and (B) to Cartesian co-ordinates by writing, as usual,

$x,\ y,\ z = x\cos\alpha + y\sin\alpha - p_1,\quad x\cos\beta + y\sin\beta - p_2,\quad x\cos\gamma + y\sin\gamma - p_3,$

where $\beta - \gamma = 180° - A$, etc.

The equation (A) thus becomes in Cartesian co-ordinates

$$F(x\cos\alpha + y\sin\alpha - p_1,\ x\cos\beta + y\sin\beta - p_2,\ x\cos\gamma + y\sin\gamma - p_3) = 0,$$

and the equation to the pair of straight lines through the origin of co-ordinates parallel to the lines (A), is

$$F(x\cos\alpha + y\sin\alpha,\ x\cos\beta + y\sin\beta,\ x\cos\gamma + y\sin\gamma) = 0 \quad \ldots \quad (A').$$

The equation in Cartesian co-ordinates to the straight line through the origin parallel to the (B) is similarly

$$\lambda x + \mu y = 0 \ \ldots \quad \ldots \quad \ldots \quad \ldots \quad (B')$$

where $\lambda,\ \mu = l\cos\alpha + m\cos\beta + n\cos\gamma,\ l\sin\alpha + m\sin\beta + n\sin\gamma$.

Now the line (B) will be perpendicular to one of the lines (A) if the line (B′) is perpendicular to one of the lines (A′). The condition that (B′) should be perpendicular to one of the lines (A′) is found by substituting in the equation (A′) λ, μ for x, y. The line (B) will therefore be perpendicular to one of the lines (A) if

$$F(\lambda\cos\alpha + \mu\sin\alpha,\ \lambda\cos\beta + \mu\sin\beta,\ \lambda\cos\gamma + \mu\cos\gamma) = 0.$$

Replacing λ, μ by their values in terms of l, m, n, we get

$$\begin{aligned} \lambda\cos\alpha + \mu\cos\beta &= \cos\alpha(l\cos\alpha + m\cos\beta + n\cos\gamma) \\ &\quad + \sin\alpha(l\sin\alpha + m\sin\beta + n\sin\gamma) \\ &= l + m\cos(\alpha\sim\beta) + n\cos(\gamma\sim\alpha) \\ &= l - m\cos C - n\cos B, \end{aligned}$$

with similar values $\lambda\cos\beta + \mu\sin\beta$ and $\lambda\cos\gamma + \mu\sin\gamma$. Hence the theorem follows.

2. Taking now the conic

$$S \equiv (a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ z)^2 = 0$$

and the straight line

$$lx + my + nz = 0 \ \ldots \quad \ldots \quad \ldots \quad (P),$$

we write down the equation to the pair of tangents to the conic at the points where the straight line cuts the conic. This equation is

$$S\Sigma = \Delta(lx + my + nz)^2 \quad \ldots \quad \ldots \quad (T),$$

where Σ is written for $(A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)^2$, and Δ, A, B, C, F, G, H have their usual meanings.

The line (P) will be a normal to the conic S if it is perpendicular to one of the lines (T). The condition for this is, by (C), found by substituting $l - m\cos C - n\cos B$, etc., for x, y, z in (T). The result is $(a,\ b,\ c,\ f,\ g,\ h)(l - m\cos C - n\cos B,\ m - n\cos A - l\cos C,\ n - l\cos B - m\cos C)^2 \times \Sigma = \Delta(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C)^2$, the condition sought for.

The triangle and its escribed parabolas.

By A. J. Pressland, M.A.

§ 1. The problem "to inflect a straight line between two sides of a triangle so that the intercepted portion is equal to the segments cut off" has been discussed in the third volume of the *Proceedings*.

If we discuss the same analytically; taking CB and CA as axes of x and y (Fig. 1) and calling each segment k, the equation of the line considered is

$$x/(a-k) + y/(b-k) = 1, \qquad \ldots \qquad \ldots \qquad (\alpha)$$

where

$$k^2 = (a-k)^2 + (b-k)^2 - 2(a-k)(b-k)\cos C \qquad \ldots \qquad (\beta)$$

The envelope of (α) considering k unrestricted by (β) is

$$(x+y)^2 - 2(a-b)(x-y) + (a-b)^2 = 0 \qquad \ldots \qquad (\gamma)$$

a parabola touching the axis of x at $(a-b, 0)$
and the axis of y at $(0, b-a)$
and which can be shown to touch AB
at the point $\left(\dfrac{a^2}{a-b},\ -\dfrac{b^2}{a-b}\right)$.

Its axis is $x + y = 0$
and tangent at vertex $x - y = \dfrac{a-b}{2}$.

§ 2. If we consider $x/(a-k) + y/(b+k) = 1$
which cuts off equal portions from BC and CA produced, the envelope is

$$(x-y)^2 - 2(a+b)(x+y) - (a+b)^2 = 0,$$

which touches CB at $(a+b,\ 0)$ the point l,
CA at $(0,\ a+b)$ the point k,
AB at $\left(\dfrac{a^2}{a+b}, \dfrac{b^2}{a+b}\right)$ the point t,
the axis being $x - y = 0$
and tangent at vertex $x + y = \dfrac{a+b}{2}$.

§ 3. The foci of these parabolas are

$$x = -y = (a-b)/4\sin^2\tfrac{1}{2}C$$
$$x = y = (a+b)/4\cos^2\tfrac{1}{2}C$$

§ 4. Hence if ABC be the triangle and we bisect the angles we get the orthic system F, D, E, O.

Let α, β, γ be mid points of sides,

a, b, c be mid points of OD, OE, OF;

then A, B, C, α, β, γ, a, b, c are on the nine point circle of DEF.

We have shown that α, β, γ, a, b, c are foci of parabolas touching all the sides of ABC, so that any tangent cuts off equal portions from two sides, viz. :—

γ and c parabolas from CB and CA,
β and b parabolas from BC and BA,
α and a parabolas from AB and AC,

the Greek letter corresponding to direct section,
the Italian letter corresponding to transverse section.

§ 5. The points of contact of the parabolas are :—

(i.) on lines through C, $(a \pm b)$ distant from C.

(ii.) on AB dividing AB internally and externally in ratio $a : b$, oppositely to the bisectors of the angle C.

Whence the four lines CT, CA, Ct, CB form an harmonic pencil;

$$\text{T is } \left(\frac{a^2}{a-b},\ -\frac{b^2}{a-b}\right) \text{ and therefore lies on } kl,$$

$$t \text{ is } \left(\frac{a^2}{a+b},\ \frac{b^2}{a+b}\right) \text{ and therefore lies on } fg.$$

§ 6. Taking any tangent so that the intercepted portion is equal to m times the segment cut off we have the equation

$$k^2(m^2 - 2 + 2\cos C) + 2k(a+b)(1 - \cos C) - c^2 = 0. \quad \ldots \quad (\delta)$$

This shows that we can get two $(1 : m : 1)$ lines.

Let these be

$$x/(a-k_1) + y/(b-k_1) = 1$$
$$x/(a-k_2) + y/(b-k_2) = 1.$$

Their point of intersection is

$$(a-b)x=(a-k_1)(a-k_2)=\frac{a^2(m^2-4\sin^2\tfrac{1}{2}\mathrm{C})+4a(a+b)\sin^2\tfrac{1}{2}\mathrm{C}-\ c}{m^2-4\sin^2\tfrac{1}{2}\mathrm{C}}$$

$$(b-a)y=(b-k_1)(b-k_2)=\frac{b^2(m^2-4\sin^2\tfrac{1}{2}\mathrm{C})+4b(a+b)\sin^2\tfrac{1}{2}\mathrm{C}-c^2}{m^2-4\sin^2\tfrac{1}{2}\mathrm{C}}$$

Eliminating m we get as the locus of such points

$$\frac{(a-b)x-a^2}{(b-a)y-b^2}=\frac{4a(a+b)\sin^2\tfrac{1}{2}\mathrm{C}-c^2}{4b(a+b)\sin^2\tfrac{1}{2}\mathrm{C}-c^2}=\frac{a(a+b)(a-b)^2/c^2-a^2}{b(a+b)(a-b)^2/c^2-b^2}$$

$$=\frac{4a^2\sin^2\tfrac{1}{2}\mathrm{C}-(a-b)^2}{4b^2\sin^2\tfrac{1}{2}\mathrm{C}-(a-b)^2}\quad\ldots\quad\ldots\quad\ldots\quad(\epsilon)$$

which show that the line passes through T, through the focus γ, and through $\left(\frac{a(a^2-b^2)}{c^2},-\frac{b(a^2-b^2)}{c^2}\right)$.

The last is a special point K; which lies not only on $bx+ay=0$, the line through the origin parallel to AB; but also on the circumscribing circle.

§ 7. The point determined by the intersection of the $\begin{Bmatrix}1:1:1\\1:1:1\end{Bmatrix}$ lines is

$$(a-b)(a^2+b^2-c^2-ab)x=(2a-b)ab^2$$
$$(b-a)(a^2+b^2-c^2-ab)y=(2b-a)a^2b\,;$$

it lies on

$$\frac{x}{(2a-b)b}+\frac{y}{(2b-a)a}=0$$

the line joining the origin to the intersection of

$$x/b-y/a=1$$

and $$x/(2a-b)-y/(2b-a)=1.$$

The corresponding point for the c – parabola is

$$x(a-b)(a^2+b^2-c^2+ab)=(2a+b)(ab^2)$$
$$y(b-a)(a^2+b^2-c^2+ab)=(2b+a)(ba^2).$$

§ 8. As (ϵ) passes through T it follows that the two $(1:m:1)$ lines being tangents are equally inclined to AB.

A similar set of theorems are true of the c – parabola, and as c and γ are mid points of the arcs AcB and AγB it follows that th loci (ϵ) and (ϵ') are at right angles.

§ 9. If we consider the $(l : m : n)$ line whose equation is

$$x/(a - lk) + y/(b - nk) = 1$$

we get a similar set of theorems; which can be verified by projection from those already found; and the loci (ϵ_1) (ϵ_1') will pass through the point K as before.

(ϵ_1) is $$\frac{(an - bl)x - a^2n}{(bl - an)y - b^2l} = \frac{a(b^2 - a^2)(an - bl) + c^2na^2}{-b(b^2 - a^2)(bl - an) + c^2b^2l}$$

and on it will be situated two special points.

(i.) When $m = n$ the point is

$$x = \frac{1}{a - b\lambda} \cdot \frac{2ab - b^2\lambda}{2\cos C - \lambda}.$$

$$y = \frac{1}{b\lambda - a} \cdot \frac{b^2 - (b\lambda - a)^2}{2\cos C - \lambda}$$

where $$\lambda = l/n.$$

Eliminating λ we get

$$(kx - b^2 + by)\left(kx - l^2 + \frac{ak}{b}y\right) = (bx + ay - ab)^2$$

where $$k = 2b\cos C - a$$

an hyperbola having one asymptote parallel to AB and the other parallel to

$$(k^2 - b^2)x + b(k - a)y = 0.$$

(ii.) When $l = m$ we get as locus an hyperbola having one asymptote parallel to AB and the other parallel to

$$\frac{ax}{(a^2 - c^2)^2 - a^2b^2} + \frac{y}{b(a^2 - b^2 - c^2)} = 0.$$

§ 10. It can be shown by transversals that Ag, Bf and CT meet in a point Q.

The locus of Q is the minimum ellipse* circumscribing the triangle ABC.

A geometrical proof of this without projections will be given in § 14.

The theorem may be stated thus:—If three parallel lines be drawn through the vertices of a triangle their isotoms intersect on the minimum ellipse.

---

* Steiner's *Gesammelte Werke*, Vol. I., p. 208.

§ 11. Different series of parabolas may be obtained by considering the envelopes of

$$\begin{aligned} x/k \quad &+ y/(b+k) = 1, \\ x/(a+k) &+ y/(k) \quad = 1, \\ x/k \quad &+ y/(b-k) = 1, \\ x/(a-k) &+ y/k \quad = 1. \end{aligned}$$

§ 12. A principle of triality would seem to hold; because any of these parabolas might be referred to A or B instead of C as origin. Such theorems can be transformed by means of the transversal property.

$$\frac{\mathrm{AE{\cdot}AF}}{\mathrm{CA{\cdot}AB}} + \frac{\mathrm{AF{\cdot}BD}}{\mathrm{AB{\cdot}BC}} + \frac{\mathrm{CD{\cdot}AE}}{\mathrm{BC{\cdot}CA}} = 0,$$

where DEF is a transversal cutting BC in D, CA in E, AB in F.

§ 13. Corresponding to K we get two other points by drawing through B and A parallels to the opposite sides. Calling these points K_c, K_b, K_a we have

$$\text{arc } AB = \text{arc } K_aC = \text{arc } K_bC.$$

Then $\qquad \angle K_aK_cK_b = \pi - K_aBK_b = \pi - 2C.$

Thus $K_aK_bK_c$ has angles $\pi - 2A$, $\pi - 2B$, $\pi - 2C$ and is similar to the pedal or orthic triangle..

Again $\qquad \text{arc } AB = \text{arc } K_bC;$

therefore $\qquad \text{arc } AK_b = \text{arc } CB,$

and $\qquad \angle AK_aK_b = \angle BAC = A.$

Now $\qquad \angle K_bK_aC = \pi - 2A;$

therefore $\qquad \angle CK_aR = A;$

therefore $\qquad K_a\alpha\gamma = A = K_a\gamma\alpha,$

and B, A, β, α are concyclic.

The sides of $K_aK_bK_c$ are therefore anti-parallel to those of ABC, and in pairs equally inclined to the sides of ABC.

The circumscribing circles of the triangles formed by $K_a\beta\alpha$, $\beta\delta C$, $K_a\delta\gamma$, $\alpha\gamma C$ meet in a point Q, say.

Let the circumcircle of $\gamma\delta C$ cut that of ABC in F;

then $\qquad \gamma\delta C = \alpha\gamma\delta - \alpha C\delta$

$\qquad\qquad = A - C;$

therefore $\qquad \gamma FC = A - C.$

Now $\qquad K_aFC = \pi - C$ for $K_aC = AB$;

therefore $\qquad K_aF\gamma = \pi - A,$

or $\qquad K_aF\gamma + K_a\alpha\gamma = \pi,$

and therefore the circumcircle of $K_a\alpha\gamma$ passes through F, and hence Q and F coincide.

If the sides of the triangles ABC, $K_aK_bK_c$ be taken three and three, 20 triangles may be obtained whose circumcircles all pass through F. Two of these circumcircles obviously coincide.

F is Steiner's point for the triangle ABC, as may be proved by analysis.

It may also be shown that C is on the radical axis of $AK_a\beta$ and $BK_b\alpha$.

The triangle $K_aK_bK_c$ is twice the linear dimensions of the orthic triangle of ABC, and in position it is this same triangle turned through two right angles.

If instead of turning the triangle $K_aK_bK_c$ through two right angles, we turn ABC through two right angles we get another position of F diametrically opposite its former one. This new position is called Tarry's point.

It should be noticed that as $K_aK_bK_c$ is derived from ABC so can ABC be derived from the median triangle of DEF.

§ 14. MINIMUM ELLIPSE.

ABC is the triangle, K the intersection of the lines AD, BE, CF, D, E, F, points of contact of parabola.

Join AG and produce to H so that $GH = GA$; let AH cut BE in M, and join KH cutting AC in L.

We shall show the anharmonic ratio.

K(ABHC) is constant; whence K is on the minimum ellipse.

Let $\lambda = \dfrac{AF}{BA} = \dfrac{AE}{CE} = \dfrac{BC}{CD}$ [Apollonius *Conics*, III. 41.]

We have by transversals

$$EK \cdot MH \cdot AL = MK \cdot AH \cdot EL,$$

$$BK \cdot AF \cdot CE = EK \cdot BF \cdot AC,$$

and $\qquad AE \cdot BC \cdot \alpha M = CE \cdot \alpha B \cdot AM$

or $\qquad 2AE \cdot \alpha M = CE \cdot AM;$

therefore $\qquad \dfrac{BK}{EK} = \dfrac{BF \cdot AC}{AF \cdot CE} = \dfrac{(\lambda+1)^2}{\lambda};$

therefore $\qquad \dfrac{BE}{EK} = \dfrac{\lambda^2+\lambda+1}{\lambda}.$

Again
$$\frac{Ca}{Ba}\cdot\frac{BM}{EM}\cdot\frac{EA}{CA}=1,$$

or
$$\frac{BM}{EM}=\frac{CA}{EA}=\frac{\lambda+1}{\lambda};$$

therefore
$$\frac{BE}{EM}=\frac{2\lambda+1}{\lambda};$$

therefore
$$\frac{EM}{EK}=\frac{\lambda}{2\lambda+1}\cdot\frac{\lambda^2+\lambda+1}{\lambda},$$
$$=\frac{\lambda^2+\lambda+1}{2\lambda+1};$$

or
$$\frac{MK}{EK}=\frac{\lambda^2+3\lambda+2}{2\lambda+1}.$$

Now
$$\frac{AM}{aM}=2\frac{AE}{CE}=2\lambda;$$

therefore
$$\frac{Aa}{aM}=\frac{2\lambda+1}{1};$$

therefore
$$aM=\frac{Aa}{2\lambda+1},$$

and
$$HM=\frac{Aa}{3}+\frac{Aa}{2\lambda+1},$$
$$=\frac{2(\lambda+2)}{3(2\lambda+1)}Aa,$$
$$=\tfrac{1}{2}\cdot\frac{\lambda+2}{2\lambda+1}\cdot AH.$$

Hence
$$\frac{HM}{AH}\cdot\frac{EK}{MK}=\tfrac{1}{2}\frac{(\lambda+2)}{(2\lambda+1)}\cdot\frac{2\lambda+1}{\lambda^2+3\lambda+2},$$
$$=\tfrac{1}{2}\frac{1}{\lambda+1};$$

therefore
$$\frac{AL}{EL}=2\frac{AE+CE}{CE};$$

whence
$$\frac{AE\cdot CL}{CE\cdot AL}=\tfrac{1}{2}; \text{ the required result.}$$

Since
$$\frac{AL}{CL}=2\frac{AE}{CE}=\frac{AM}{aM},$$

ML is parallel to BC.

It may be noticed that GL passes through mid point of CD.

Some Geometrical Theorems.

By R. E. Allardice, M.A.

1. The greatest line joining two points in the perimeter of a polygon is a side or a diagonal of the polygon.

For (fig. 2) PS is obviously less than one or other of the lines PQ, PR. Hence if AB, CD (fig. 3) are any two finite lines, L and M any points, one in each of these lines, LM is less than LC, or LD and LC is less than CA or CB. From this the theorem follows at once. The theorem and the above proof apply to crossed and gauche polygons.

2. Let ABC (fig. 4) be any triangle; CQ equal to CB and perpendicular to it; CP equal to CA and perpendicular to it; then the locus of O, their intersection, is a circle on AB.

The proof of this theorem is very simple; but what is remarkable about the theorem is that the point C has two degrees of freedom; and it would seem at first sight as if O ought to have two degrees of freedom also.

Since to a two-fold infinity of positions of C corresponds a one-fold infinity of positions of O, it is to be expected that to every position of O will correspond a one-fold infinity of positions of C; that is, C will have a locus. This locus is a straight line passing through O and through the middle point of the semicircle described on AB on the side remote from C; as may easily be shown.

3. A PROPERTY OF THE ELLIPSE.

If (x, y, z) be the trilinear co-ordinates of a point P, it may easily be shown that $x^2+y^2+z^2$ is a minimum when P is at (a, b, c), which is the symmedian point.

Now if P (fig. 5) move so that y^2+z^2 is constant, the locus of P is an ellipse of which AB and AC are the equi-conjugate diameters. But for the minimum position PD must be normal to the conic; and thus the tangent at P, which is the symmedian point, is parallel to BC.

Hence we get the following theorem :—

If the tangent to an ellipse at the point T (fig. 6) meets the equi-conjugate diameters in Q and Q′, then CT is a symmedian of the triangle CQQ′.

This theorem may also be proved directly as follows:—Let the tangent at T meet the axes in P and P′. Now it is known that pairs of conjugate diameters trace out on the tangent at T an involution, of which T is the centre.

Let R be the middle point of QQ′, and O the middle point of PP′. We have to show that P′TPR is a harmonic range; that is, $(P'TPR) = -1$.

We have $TP.TP' = TQ.TQ'$ and $(P'QPQ') = -1$.

Now
$$\begin{aligned} TQ.TQ' &= (OT - OQ)(OQ' - OT) \\ &= OT(OQ + OQ') - OQ.OQ' - OT^2 \\ &= 2OT.OR - OP^2 - OT^2. \end{aligned}$$

But $TQ.TQ' = TP.TP' = OP^2 - OT^2$.

Hence $OP^2 - OT^2 = 2OT.OR - OP^2 - OT^2$

$$\therefore \quad OP^2 = OT.OR.$$

Hence P′TPR is a harmonic range; and, since P′OP is a right angle, OP bisects TCR.

The above proof contains a proof of the following property of an involution:—

If PP′, QQ′ are pairs of corresponding points in an involution of which T is the centre; and if P and P′ are harmonically conjugate with reference to Q and Q′; then, if O is the middle point of PP′ and R the middle point of QQ′, O is harmonically conjugate to T with reference to Q and Q′, and R is harmonically conjugate to T with reference to P and P′.

4. The following simple geometrical proof of a proposition given by Mr Pressland in his paper *On the triangle and its escribed parabolas*, contained in this volume, may be worth recording.

Let ABC (fig. 7) be the triangle; BP equal to CQ; to show that the envelope of PQ is a parabola touching all the sides of the triangle. (BP and CQ may be taken on the same side or on opposite sides of BC, and may be taken in any constant ratio, and the following proof, with some slight modification, will still apply).

The proof depends on the following converse of a well-known property of the parabola:—

If there be two straight lines OX and OY and a fixed point F, and if a circle through O and F meets OX and OY in P and Q, then PQ envelopes a parabola which touches OX and OY and of which F is the focus.

Let O be the circumcentre of the triangle ABC, and R the circumcentre of APQ. Then RO has equal projections on AB and AC (each being $\frac{1}{2}$BP); and hence RO is parallel to the bisector of the angle BAC. Therefore the locus of R is a straight line through O, parallel to the bisector of the angle BAC; and thus the circumcircle of APQ always passes through a fixed point A′, which is the image of A in RO. Hence, by the proposition stated above, PQ envelopes a parabola of which A′ is the focus, and which touches AB and AC.

It is obvious from the above also that if any circle through A and A′ meets AB and AC in P and Q, then BP = BQ; and the following more general theorem may also be proved very easily:—

If a series of circles pass through two fixed points A and A′, and if two straight lines APQR ... and AP′Q′R′ ... meet these circles in the points P, Q, R, ... and P′, Q′, R′, ..., then PQ : QR : ... = P′Q′ : Q′R′ : ....

This theorem may also be got by inversion; it may in fact be got by inverting the theorem that the anharmonic ratio of a pencil is constant, by taking the particular case of a number of concurrent transversals.

Third Meeting, January 9th, 1891.

R. E. Allardice, Esq., President, in the Chair.

On the Heating of Conductors by Electric Currents, and the Electric Distribution in Conductors so heated.

By John M'Cowan, M.A., B.Sc.

The solution of the equation.

$$\sqrt{\{(x-a)(x-b)\}} + \sqrt{\{(x-c)(x-d)\}} = e.$$

By J. D. Hamilton Dickson, M.A.

1. If $P = a + b - c - d$, and $K = cd - ab$, the solution of the equation

$$\sqrt{\{(x-a)(x-b)\}} + \sqrt{\{(x-c)(x-d)\}} = e \quad \ldots \quad (1)$$

may be most readily found by putting $(x-a)(x-b) = z^2$; whence

$(x-c)(x-d) = z^2 + \mathrm{P}x + \mathrm{K}$, and therefore $= z^2 - 2ez + e^2$, and finally x is given by the equation

$$(\mathrm{P}^2 - 4e^2)x^2 + 2\{\mathrm{P}(\mathrm{K} - e^2) + 2e^2(a+b)\}x + (\mathrm{K} - e^2)^2 - 4e^2ab = 0 \qquad (2)$$

In order to simplify the discussion of this result I use the following notation :—

$$\begin{array}{lll}
\mathrm{P} = a + b - c - d, & \alpha = (a-c)(a-d), & \kappa = (a-b)(c-d), \\
\mathrm{Q} = a - b + c - d, & \beta = (b-c)(b-d), & \lambda = (a-c)(b-d), \\
\mathrm{R} = a - b - c + d, & \gamma = (c-a)(c-b), & \mu = (a-d)(b-c)\,; \\
 & \delta = (d-a)(d-b), & \\
\Pi = \mathrm{P}^2 - 4e^2, & \mathrm{L} = e^2 - \lambda, & \mathrm{M} = e^2 - \mu.
\end{array}$$

The solution of (2) may now be written in the form

$$\Pi x = -\mathrm{P}(\mathrm{K} - e^2) - 2e^2(a+b)$$
$$\pm \sqrt{[\{\mathrm{P}(\mathrm{K} - e^2) + 2e^2(a+b)\}^2 - \Pi\{(\mathrm{K} - e^2)^2 - 4e^2ab\}]},$$

which, after some simplifications, becomes

$$\Pi x = -e^2\{-\mathrm{P} + 2(a+b)\} - \mathrm{PK} \pm 2e\sqrt{(\mathrm{LM})} \qquad (3)$$

whence

$$\left.\begin{array}{l}
\Pi(x-a) = e^2(\ \ \mathrm{P} + \mathrm{Q} + \mathrm{R}) - \mathrm{P}\alpha \pm 2e\sqrt{(\mathrm{LM})}, \\
\Pi(x-b) = e^2(\ \ \mathrm{P} - \mathrm{Q} - \mathrm{R}) - \mathrm{P}\beta \pm 2e\sqrt{(\mathrm{LM})}, \\
\Pi(x-c) = e^2(-\mathrm{P} + \mathrm{Q} - \mathrm{R}) + \mathrm{P}\gamma \pm 2e\sqrt{(\mathrm{LM})}, \\
\Pi(x-d) = e^2(-\mathrm{P} - \mathrm{Q} + \mathrm{R}) + \mathrm{P}\delta \pm 2e\sqrt{(\mathrm{LM})},
\end{array}\right\} \quad \cdots \quad (4)$$

2. It is worth while verifying that the results in (4) satisfy (1). Multiplying together the first two,

$$\begin{aligned}
\Pi^2(x-a)(x-b) = {} & e^4\{\mathrm{P} + 2(a-b)\}\{\mathrm{P} - 2(a-b)\} \\
& - e^2\mathrm{P}[\{\mathrm{P} + 2(a-b)\}\beta + \{\mathrm{P} - 2(a-b)\}\alpha] \\
& + \mathrm{P}^2\alpha\beta + 4e^2\mathrm{LM} \pm 2e\{2\mathrm{P}e^2 - \mathrm{P}(\alpha+\beta)\}\sqrt{(\mathrm{LM})} \\
= {} & 4e^6 + e^4\{\mathrm{P}^2 - 4(a-b)^2 - 4\lambda - 4\mu\} \\
& + e^2\{-\mathrm{P}^2(\alpha+\beta) + 2\mathrm{P}(a-b)(\alpha-\beta) + 4\lambda\mu\} \\
& + \mathrm{P}^2\alpha\beta \pm 2\{2e^3 - (\alpha+\beta)e\}\mathrm{P}\sqrt{(\mathrm{LM})}
\end{aligned}$$

or, noting that $\lambda\mu = \alpha\beta$, and that $\alpha - \beta = \mathrm{P}(a-b)$, and making some further reductions,

$$\begin{aligned}
\Pi^2(x-a)(x-b) = {} & 4e^6 - 4(\alpha+\beta)e^4 + (\alpha+\beta)^2e^2 \\
& + \mathrm{P}^2e^4 - \mathrm{P}^2(\lambda+\mu)e^2 + \mathrm{P}^2\lambda\mu \pm 2\{2e^3 - (\alpha+\beta)e\}\mathrm{P}\sqrt{(\mathrm{LM})},
\end{aligned}$$

and therefore,

$$\pm\Pi\sqrt{\{(x-a)(x-b)\}} = 2e^3 - (\alpha+\beta)e \pm \mathrm{P}\sqrt{(\mathrm{LM})}. \quad \cdots \quad (5)$$

Similarly,

$$\pm\Pi\sqrt{\{(x-c)(x-d)\}} = 2e^3 - (\gamma+\delta)e \mp \mathrm{P}\sqrt{(\mathrm{LM})}\,; \quad \cdots \quad (6)$$

whence,

$$\begin{aligned}\Pi[\sqrt{\{(x-a)(x-b)\}}+\sqrt{\{(x-c)(x-d)\}}] &= 4e^3-(\alpha+\beta+\gamma+\delta)e,\\ &= \pm\Pi e,\end{aligned}$$

since $\alpha+\beta+\gamma+\delta=P^2$.

3. This verification suggests the possibility of $\Pi(x-a)$ being also expressible in the form of a square. In the case of $A+\sqrt{C}$ being the same as $(\sqrt{u}+\sqrt{v})^2$, and $B+\sqrt{C}$ being the same as $(\sqrt{u'}+\sqrt{v'})^2$, it follows at once that $(A+\sqrt{C})(B+\sqrt{C})$ is of a similar form. But the converse is not true. The necessary condition is that $(A^2-C)(B^2-C)$ be a complete square, which is satisfied if

$$A^2-C=F^2\Theta,\ \text{and}\ B^2-C=G^2\Theta.$$

To examine the case for $\pi(x-a)$, I look for the factors of

$$\{e^2(P+Q+R)-Pa\}^2-4e^2LM.$$

Changing the sign of this expression, for a moment, it

$$\begin{aligned}&=4e^2(e^2-\lambda)(e^2-\mu)-[\{P+2(a-b)\}e^2-Pa]^2\\ &=4e^6-e^4\{P^2+4P(a-b)+4(a-b)^2+4\lambda+4\mu\}\\ &\qquad+e^2\{2P^2a+4Pa(a-b)+4\lambda\mu\}-P^2a^2\\ &=4e^6-e^4(P^2+8a)+e^2(2P^2a+4a^2)-P^2a^2\\ &=(4e^2-P^2)(e^2-a)^2\\ &=-\Pi(e^2-a)^2.\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\qquad(7)\end{aligned}$$

Hence

$$\begin{aligned}\Pi(x-a)&=e^2(P+Q+R)-Pa\pm 2e\sqrt{(LM)}\\ &=[\sqrt{\{\tfrac{1}{2}(P+Q+R)e^2+\tfrac{1}{2}(e^2-a)\sqrt{\Pi}\}}\\ &\qquad+\sqrt{\{\tfrac{1}{2}(P+Q+R)e^2-\tfrac{1}{2}(e^2-a)\sqrt{\Pi}\}}]^2,\qquad(8)\end{aligned}$$

and similarly for the other expressions in (4).

The usefulness of such a form as (8) depends on its freedom from radicals. Two of such cases may be noted. Let $e^2=\lambda$,

then $\Pi=P^2-4\lambda=(a-b-c+d)^2=R^2$

and $\lambda-a=-(a-b)(a-c)$,

whence

$$\begin{aligned}x-a&=-(a-b)(a-c)/R,\\ x-b&=-(a-b)(b-d)/R,\\ x-c&=-(a-c)(c-d)/R,\\ x-d&=-(b-d)(c-d)/R.\end{aligned}$$

Similarly, when $e^2=\mu$, we have $\Pi=Q^2$, and

$$\begin{aligned}x-a&=-(a-b)(a-d)/Q,\\ x-b&=-(a-b)(b-c)/Q,\\ x-c&=-(b-c)(c-d)/Q,\\ x-d&=\ \ (a-d)(c-d)/Q.\end{aligned}$$

These are what we get from the vanishing of L and M in the general solution. Other four cases are presented in this form of the solution, viz., when $e^2 = \alpha$, β, γ, δ; but these will be referred to further on, under a new aspect.

4. Thus treated, the complete solution of the general equation is given in terms of e. A case of more than usual interest arises when $\Pi = 0$, or $P^2 = 4e^2$. The equation then takes one of two forms, either

$$\text{(I.)}\quad 2\sqrt{\{(x-a)(x-b)\}} + 2\sqrt{\{(x-c)(x-d)\}} = a + b - c - d$$
$$= -(x-a) - (x-b) + (x-c) + (x-d),$$

$$\left.\begin{array}{ll} \text{whence} & \sqrt{(x-a)} + \sqrt{(x-b)} + \sqrt{(x-c)} - \sqrt{(x-d)} = 0, \\ \text{or} & \sqrt{(x-a)} + \sqrt{(x-b)} - \sqrt{(x-c)} + \sqrt{(x-d)} = 0; \end{array}\right\} \qquad (9)$$

$$\text{or (II.)}\quad 2\sqrt{\{(x-a)(x-b)\}} + 2\sqrt{\{(x-c)(x-d)\}} = -a - b + c + d,$$

$$\left.\begin{array}{ll} \text{whence} & \sqrt{(x-a)} - \sqrt{(x-b)} + \sqrt{(x-c)} + \sqrt{(x-d)} = 0, \\ \text{or} & -\sqrt{(x-a)} + \sqrt{(x-b)} + \sqrt{(x-c)} + \sqrt{(x-d)} = 0. \end{array}\right\} \qquad (9')$$

The solution is got from (2) at once, in the form

$$x = \frac{4e^2ab - (K - e^2)^2}{4e^2(a+b) + 2P(K - e^2)},$$

or,

$$x = \{P^2ab - (K - \tfrac{1}{4}P^2)\}/\tfrac{1}{2}PQR \quad \ldots \quad \ldots \quad (10)$$

whence

$$\left.\begin{array}{l} x - a = -(e^2 - \alpha)^2/\tfrac{1}{2}PQR, \\ x - b = -(e^2 - \beta)^2/\tfrac{1}{2}PQR, \\ x - c = -(e^2 - \gamma)^2/\tfrac{1}{2}PQR, \\ x - d = -(e^2 - \delta)^2/\tfrac{1}{2}PRR. \end{array}\right\} \quad \ldots \quad \ldots \quad (11)$$

These results agree with equations (9); for, irrespective of sign, sum of square roots of numerators $= 4e^2 - (\alpha + \beta + \gamma + \delta) = 0$.

The results in (11) are remarkable as containing the complete and (as will appear afterwards) the unique solution of the following problem in Diophantine Analysis :—

To find integral values for p *and* q *to satisfy the conditions that* ap - q, bp - q, cp - q, dp - q *are all positive squares.*

The solution holds if any one, two, or three of the quantities a, b, c, d is zero ; they may be positive or negative : but obviously the solution fails if P, Q, or R vanish.

5. Further insight into the properties of the equation considered may be obtained by geometry.

Let O, A, B, C, D (fig. 8), be five points (in order) on a straight line, and if a, b, c, d be in ascending order of magnitude they may be taken as the numerical values of the length OA, OB, OC, OD. If x be the corresponding abscissa of some point X on the line OD, then, for e to be real, obviously X must lie outside of both the circles described on AB and CD as diameters. Let G, H be the centres, respectively, of these circles; and let the common tangents EF, E′F′ cut the line GH (produced if necessary) in X_1, X_1' respectively: then X_1, X_1' are the centres of similitude of these circles. Hence we find,

$$\left.\begin{array}{rlrl} GH = OH - OG = & -\tfrac{1}{2}P, & OX_1 = & (ad-bc)/R, \\ FH - EG = & \tfrac{1}{2}R, & OX_1' = & (ac-bd)/Q \\ F'H + E'G = & -\tfrac{1}{2}Q, & X_1X_1' = & P\kappa/QR. \end{array}\right\} \quad \ldots \quad (12)$$

Now, let us suppose $x = OX_1$, then

$$\left.\begin{array}{l} X_1E = \surd\{(x-a)(x-b)\} = (a-b)\surd\lambda/R \\ X_1F = \surd\{(x-c)(x-d)\} = (c-d)\surd\lambda/R \end{array}\right\} \quad \ldots \quad (13)$$

$$\left.\begin{array}{lll} \text{whence} & e = X_1E + X_1F = & Q\surd\lambda/R \\ \text{and} & e = -X_1E + X_1F = & -\surd\lambda \end{array}\right\} \quad \ldots \quad \ldots \quad (14)$$

Again, if $x = OX_1'$, then

$$X_1'E' = (a-b)\surd\mu/Q, \quad X_1'F' = (c-d)\surd\mu/Q, \quad \ldots \quad (15)$$

$$\left.\begin{array}{lll} \text{whence} & e = X_1'E' + X_1'F' & = \surd\mu \\ \text{and} & e = -X_1'E' + X_1'F' & = -R\surd\mu/Q \end{array}\right\} \quad \ldots \quad (16)$$

6. Let, next, the numerical quantities a, b, c, d be arranged in a new order, under the signs of the square root—viz., the first and third under one root, and the second and fourth under the other root, so that the equation is now

$$\surd\{(x-a)(x-c)\} + \surd\{(x-b)(x-d)\} = e.$$

For a real value of e, X must lie outside of the circles (Fig. 9) described on AC, and BD as diameters. Here, again, we find the values,

$$\left.\begin{array}{rlrl} GH = OH - OG = & -\tfrac{1}{2}Q, & OX_2 = & (ad-bc)/R, \\ FH - EG = & \tfrac{1}{2}R, & OX_2' = & (ab-cd)/P, \\ FH + EG = & -\tfrac{1}{2}P, & X_2X_2' = & Q\lambda/PR; \end{array}\right\} \quad \ldots \quad (17)$$

and if $x = OX_2$, then

$$\left.\begin{array}{l} X_2E = \surd\{(x-a)(x-c)\} = (a-c)\surd\kappa/R, \\ X_2F = \surd\{(x-b)(x-d)\} = (b-d)\surd\kappa/R; \end{array}\right\} \quad \ldots \quad (18)$$

$$\left.\begin{array}{lll} \text{whence,} & e = X_2E + X_2F = & P\surd\kappa/R, \\ \text{and} & e = -X_2E + X_2F = & -\surd\kappa \end{array}\right\} \quad \ldots \quad \ldots \quad (19)$$

Also, if $x = OX_2'$, then

$$X_2'E' = i(a-c)\sqrt{\mu}/P, \quad X_2'F' = i(b-d)\sqrt{\mu}/P\,; \quad \ldots \quad (20)$$

$$\left.\begin{array}{lll} \text{whence} \quad e = X_2'E' + X_2'F' = & i\sqrt{\mu}, & \text{or } e^2 = -\mu, \\ \text{and} \quad e = -X_2'E' + X_2'F' = & -iR\sqrt{\mu}/P, & \text{or } e^2 = -R^2\mu/P^2. \end{array}\right\} \quad (21)$$

7. Finally, arranging the quantities a, b, c, d, so that the equation takes the form $\sqrt{\{(x-a)(x-d)\}} + \sqrt{\{(x-b)(x-c)\}} = e$; in order that e may be real, X must lie outside of the circle (fig. 10), described on AD as diameter. As before we find

$$\left.\begin{array}{ll} GH = OH - OG = -\tfrac{1}{2}R, & OX_3 = (ac - bd)/Q, \\ FH - EG = \tfrac{1}{2}Q, & OX_3' = (ab - cd)/P, \\ FH + EG = -\tfrac{1}{2}P\,; & X_3X_3' = R\mu/PQ. \end{array}\right\} \quad (22)$$

If $x = OX_3$; then

$$\left.\begin{array}{l} X_3E = \sqrt{\{(x-a)(x-d)\}} = i(a-d)\sqrt{\kappa/Q} \\ X_3F = \sqrt{\{(x-b)(x-c)\}} = i(b-c)\sqrt{\kappa/Q} \end{array}\right\} \quad \ldots \quad (23)$$

$$\left.\begin{array}{lll} \text{whence,} \quad e = X_3E + X_3F = & iP\sqrt{\kappa/Q}\,; & \text{or, } e^2 = -P^2\kappa/Q^2\,; \\ \text{and,} \quad e = -X_3E + X_3F = & -i\sqrt{\kappa}\,; & \text{or, } e^2 = -\kappa. \end{array}\right\} \quad (24)$$

Also, if $x = OX_3'$, we get

$$X_3'E' = i(a-d)\sqrt{\lambda/P}\,; \quad X_3'F' = i(b-c)\sqrt{\lambda/P} \quad \ldots \quad (25)$$

whence,

$$\left.\begin{array}{lll} e = X_3'E' + X_3'F' = & i\sqrt{\lambda}\,; & \text{or, } e^2 = -\lambda : \\ \text{and} \quad e = -X_3'E' + X_3'F' = & -iQ\sqrt{\lambda/P}\,; & \text{or, } e^2 = -Q^2\lambda/P^2. \end{array}\right\}\text{*} \quad (26)$$

The above results lead to several geometrical theorems, *e.g.*:—

If A, B, C, D be four points in order on a straight line, and if pairs of circles be described on AB, CD as diameters; on AC, BD as diameters; and on AD, BC as diameters: the three corresponding pairs of centres of similitude will be represented by only three points.

The length of the common external tangent to the circles on AB

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[*] The results of §§ 6, 7, might have been derived by substitution from those of § 5. Thus we get § 6 by the substitution $\begin{matrix} b, & c \\ c, & b \end{matrix}$, which gives

$$\begin{matrix} P, Q, R & \text{and} & \kappa, \lambda, \mu \\ Q, P, R & & \lambda, \kappa, -\mu \end{matrix} :$$

and § 7 by the substitution $\begin{matrix} b, & c, & d \\ d, & b & c, \end{matrix}$, which gives

$$\begin{matrix} P, Q, R & \text{and} & \kappa, \lambda, \mu \\ R, P, Q & & \mu, -\kappa, -\lambda \end{matrix} .$$

and CD is equal to one of the intercepts on the chord, perpendicular to AD, made by the circles on AD and BC.

8. The properties of the equation may also be studied with advantage in the form

$$\sqrt{\{(x-a)(x-b)\}}+\sqrt{\{(x-c)(x-d)\}}=y.$$

Figs 11, 12, 13 represent the equations

$$\begin{aligned}&\mp\sqrt{\{(x-1)(x-2)\}}+\sqrt{\{(x-5)(x-7)\}}=y,\\&\mp\sqrt{\{(x-1)(x-5)\}}+\sqrt{\{(x-2)(x-7)\}}=y,\\\text{and}\qquad&\mp\sqrt{\{(x-1)(x-7\})}+\sqrt{\{(x-2)(x-5)\}}=y,\end{aligned}$$

the upper sign referring to the *full* lines, the lower to the *dotted* lines.

For the upper signs, $\pm y=\frac{1}{2}\text{P}$ are the asymptotes; and, for the lower signs, $\pm y=2\left(x-\frac{a+b+c+d}{4}\right)$ are the symptotes: viz,

in fig. 11, $\pm y=4\frac{1}{2}$, and $\pm y=2(x-3\frac{3}{4})$;
in fig. 12, $\pm y=1\frac{1}{2}$, and $\pm y=2(x-3\frac{3}{4})$;
in fig. 13, $\pm y=\ \frac{1}{2}$, and $\pm y=2(x-3\frac{3}{4})$.

In fig. 11 the curve consists of three separate portions lying in the regions (1) from $x=-\infty$ to $x=1$; (2) from $x=2$ to $x=5$; and (3) from $x=7$ to $x=\infty$. In the first branch there is a point of inflexion at $x=-6{\cdot}2\ldots$, and in the second there is another point of inflexion at $x=3{\cdot}2\ldots$, both points being given by the equation

$$(1-m)x^2-\{(a+b)-m(c+d)\}x+ab-mcd=0 \qquad \ldots \quad (27)$$

where $m=(a-b)^3/(c-d)^3$. There is a minimum value of y in the *full* curve, at $x=-3$ (viz., $y=\sqrt{\lambda}=\sqrt{20}$); and a maximum value of y, in the *dotted* curve, at $x=3$ (viz., $y=\sqrt{\mu}=\sqrt{18}$).

In fig. 12 the middle portions of both curves have disappeared (become imaginary). There is a point of inflexion at $x=-5{\cdot}47\ldots$; a minimum value of y at $x=-3$ (viz., $y=\sqrt{\kappa}=\sqrt{2}$). The other point of inflexion and the maximum point have disappeared.

In fig. 13 the *full* curve, in its right branch, changes from the negative to the positive region of y; and the maximum and minimum points have disappeared as well as the points of inflexion.

9. Thus I prove that, if a, b, c, d are in ascending order of magnitude (algebraical), the equation

$$\mp \sqrt{\{(x-a)(x-b)\}} + \sqrt{\{(x-c)(x-d)\}} = e$$

has *two real* roots in the following cases,

(1) for the upper sign, if e lie between $\sqrt{\lambda}$ and $-\tfrac{1}{2}\mathrm{P}$;
(2) for the lower sign, if e lie between $\sqrt{\beta}$ and $\sqrt{\mu}$, or $\nless \sqrt{\delta}$;
(3) one for each sign, if e lie between $\sqrt{\lambda}$ and $\sqrt{\beta}$:

that it has *one real* root, and only one, in the following cases,

(1) for the upper sign, if e lie between $-\sqrt{\delta}$ and $\tfrac{1}{2}\mathrm{P}$,
between $-\sqrt{\gamma}$ and $+\sqrt{\gamma}$,
or between $-\tfrac{1}{2}\mathrm{P}$ and $\sqrt{a}$;
(2) for the lower sign, if e lie between $\sqrt{a}$ and $\sqrt{\delta}$:

and, finally, that the roots are *imaginary*,

if e be $< -\sqrt{\delta}$,
or if e lie between $\tfrac{1}{2}\mathrm{P}$ and $-\sqrt{\gamma}$,
or between $\sqrt{\mu}$ and $\sqrt{\lambda}$.

The critical cases, when $e^2 = \lambda$, μ, a, β, γ, or δ are too simple to require consideration.

The equations

$$\mp \sqrt{\{(x-a)(x-c)\}} + \sqrt{\{(x-b)(x-d)\}} = e,$$

and

$$\mp \sqrt{\{(x-a)(x-d)\}} + \sqrt{\{(x-b)(x-c)\}} + e,$$

may be discussed in a similar manner, and lead to similar, but simpler, results.

10. On examining the circumstances of the numerical example considered, § 8, it is found to belong to the class which, assuming a, b, c, d to be in ascending order of magnitude, may be characterised by the inequalities

$$\delta > a > \beta > \gamma$$

a class which is completely determined by the condition that R *is positive.* When R is negative, the inequalities are

$$a > \delta > \gamma > \beta,$$

which is the only other class that arises. By changing the sign of x, and correspondingly re-arranging the values to be assigned to a, b, c, d, this class is reduced to the former class.

Note also that λ is always greater than μ.

11. The circumstances of the roots of each equation may be suitably exhibited in the following tables :—

Fig. 11.

	Full Curve.	Dotted Curve.
	0	2
$\sqrt{\delta}$		
	0	1
$\sqrt{a}$		
	1	0
$-\frac{1}{2}P$		
	2	0
$\sqrt{\lambda}$		
	0	0
$\sqrt{\mu}$		
	0	2
$\sqrt{\beta}$		
	1	1
$\sqrt{\gamma}$		
	1	0
$-\sqrt{\gamma}$		
	0	0
$\frac{1}{2}P$		
	1	0
$-\sqrt{\delta}$		
	0	0

Fig. 12.

	Full Curve.	Dotted Curve.
	0	2
$\sqrt{\delta}$		
	0	1
$\sqrt{a}$		
	1	0
$-\frac{1}{2}P$		
	2	0
$\sqrt{\lambda}$		
	0	0
$\frac{1}{2}P$		
	1	0
$-\sqrt{\delta}$		
	0	0

Fig. 13.

	Full Curve.	Dotted Curve.
	0	2
$\sqrt{\delta}$		
	1	1
$\sqrt{a}$		
	2	0
$-\frac{1}{2}P$		
	1	0
$\frac{1}{2}P$		
	0	0

For example, in the case of an equation of the second type, if e lies between $-\sqrt{\delta}$ and $\frac{1}{2}P$, there is one root for the *full* curve whose equation is

$$-\sqrt{\{(x-a)(x-c)\}}+\sqrt{\{(x-b)(x-d)\}}=e$$

and no root for the *dotted* curve, whose equation is

$$\sqrt{\{(x-a)(x-c)\}}+\sqrt{\{(x-b)(x-d)\}}=e,$$

a, b, c, d being in ascending order of magnitude.

12. In solving equations of this kind it will be convenient to employ the following nomenclature :—the numbers a, b, c, d may be

called the *abscissæ* of the equation, a being, in particular, the *origin*: α, β, γ, δ (in preference to their square roots) may be called the *terminals* of the equation. To aid in simplifying the work, if in the given equation R is negative, it will be advisable to change the sign of x before appointing the values of the abscissæ. It will also shorten the work slightly to arrange that $a=0$, in which case b, c, d will be positive.

Example.—In the following equation, written down at random,

$$\sqrt{\{(x+5)(x-2)\}} - \sqrt{\{(x-1)(x-4)\}} = 6,$$

$R = -4$. In order to make R positive and have the origin zero, writing $-x+4$ for x, the equation reduces to the second typical form, viz.,

$$-\sqrt{\{x(x-3)\}} + \sqrt{\{(x-2)(x-9)\}} = 6,$$

in which we have the abscissæ,

$$\left.\begin{array}{l} a=0\\ b=3\\ c=2\\ d=9\end{array}\right\}\text{ whence } \begin{array}{rr} P= & -8,\\ Q= & -10,\\ R= & 4,\\ & \end{array} \quad \begin{array}{rr} \alpha= & 18,\\ \beta= & -6,\\ \gamma= & -2,\\ \delta= & 54,\end{array} \quad \begin{array}{rr} \lambda= & 12,\\ \mu= & -9,\\ & \\ & \end{array} \quad \begin{array}{rr} L= & 24,\\ M= & 45,\\ \Pi= & -80;\\ & \end{array}$$

and the asymptotes

$$y = \pm 4, \text{ and } y = \pm(2x-7).$$

As e is greater than $\sqrt{\alpha}$ [viz., $6 > \sqrt{18}$], there are no real roots of the equation. But, as e is also less than $\sqrt{\delta}$, the equation

$$\sqrt{\{x(x-3)\}} + \sqrt{\{(x-2)(x-9)\}} = 6$$

has one real root, viz., by (4)

$$\begin{aligned} x &= 4\tfrac{1}{2} \pm (9\sqrt{30})/10 = 4{\cdot}5 \pm 4{\cdot}9295\ldots\\ &= 9{\cdot}4295\ldots \text{or} - {\cdot}4295\ldots \end{aligned}$$

whence, a reference to the figure (fig. 12) shows that $-{\cdot}4295\ldots$ is the required root, $9{\cdot}4295\ldots$ being the real root of the equation

$$-\sqrt{\{x(x-3)\}} + \sqrt{\{(x-2)(x-9)\}} = -6,$$

13. The following is worthy of note; taking the two equations

$$-\sqrt{\{(x-a)(x-b)\}} + \sqrt{\{(x-c)(x-d)\}} = e$$

and $$\sqrt{\{(x-a)(x-b)\}} + \sqrt{\{(x-c)(x-d)\}} = f$$

then $$-(x-a)(x-b) + (x-c)(x-d) = ef,$$

or $$Px + K = ef.$$

The same value of x being a root of both equations we have the following theorem :—A root of the equation

$$-\sqrt{\{(x-a)(x-b)\}}+\sqrt{\{(x-c)(x-d)\}}=e$$

is also a root of the equation

$$\sqrt{\{(x-a)(x-b)\}}+\sqrt{\{(x-c)(x-d)\}}=\{(a+b-c-d)x-ab+cd\}/e \quad (28)$$

or, elimitating x by means of (4) from the right-hand side of this last equation, it is a root of the equation

$$\begin{aligned}&\sqrt{\{(x-a)(x-b)\}}+\sqrt{\{(x-c)(x-d)\}}\\&\qquad=e\{\mathrm{P}(\mathrm{P}+\mathrm{Q}+\mathrm{R})-4a\}/\Pi\pm 2\mathrm{P}\sqrt{(\mathrm{LM})}/\Pi,\end{aligned} \quad (29)$$

the ambiguity of sign being properly chosen.

Example.—We have seen that $x=9{\cdot}4295\ldots$is the root of the equation

$$-\sqrt{\{x(x-3)\}}+\sqrt{\{(x-2)(x-9)\}}=-6\,;$$

whence we find that the same value of x is the root of the equations

$$\sqrt{\{x(x-3)\}}+\sqrt{\{(x-2)(x-9)\}}=\tfrac{4}{3}x-3,$$

and

$$\sqrt{\{x(x-3)\}}+\sqrt{\{(x-2)(x-9)\}}=\tfrac{3}{5}(5+2\sqrt{30}).$$

Fourth Meeting, February 13, 1891.

R. E. Allardice, Esq., M.A., F.R.S.E., President, in the Chair.

A problem in the theory of numbers.

By T. Hugh Miller, M.A.

Let it be required to find the square integral numbers which added to a given integer shall produce a square integer, and the smallest such number.

Let N be the given number, and let the sum of N and x^2 be y^2, where x and y are integers.

Then $\mathrm{N}=y^2-x^2=(y+x)(y-x)$.

Now resolve N into its prime factors, let these $l^a\,m^b\,n^c\ldots$. Then if $y+x$ is put equal to the product of any number of these factors, and $y-x$ equal to the product of all the others, a series of values of y is obtained.

If the factor 2 is one of the factors of N, and of the first degree, then 2 must be a factor of $y+x$ or of $y-x$ only. Therefore y and x cannot be integers. Thus if N is double of an odd number it is impossible to find a solution. If 2 is one factor it must be of at least the second degree.

If N is a square number, $x=0$ is obviously one solution, and the least.

Excluding these values, we may put $x+y$ equal to a series of values the number of which is

$$1+a+b+c\ldots\quad +ab+ac+bc+\ldots\quad +abc+\ldots\,.$$

Now one half of these values are greater than $\sqrt{N}$, and since $y+x$ is always greater than $y-x$, the number of solutions is

$$\tfrac{1}{2}(1+a)(1+b)(1+c)\ldots\,.$$

Since a, b, c are by hypothesis not all even numbers, this number is of course an integer.

If the least value of x is required, $y+x$ and $y-x$ must differ by as small a number as possible. Then $y+x$ must be taken equal to the product of those prime factors of N, which differs from $\sqrt{N}$ by as small a number as possible, and is greater than $\sqrt{N}$.

For example. It is impossible to add a square number to 6 so that the sum shall be a square.

Let the given number be 525.

Now $525=3.5^2.7$, thus putting $y+x=5^2$, and $y-x=21$, we get $x=2$, $y=23$.

$$\therefore \qquad 525+2^2=23^2.$$

There are in all $\frac{1}{2}\times 2\times 3\times 2$, that is 6 solutions; the other numbers to be added being

$$10^2,\ 34^2,\ 50^2,\ 86^2,\ 262^2.$$

If 2^a is one of the factors of N, since 2 must be a factor of $y+x$, and of $y-x$, the number of solutions is

$$\tfrac{1}{2}(1+a-2)(1+b)(1+c)\ldots\ldots\,.$$

For example, let $N=1000$, that is $2^3.5^3$. There are $\frac{1}{2}(3-1)(1+3)$ that is 4 solutions. It is sufficient to consider the factors 2.5^3, as 2 must be a factor of $y+x$ and $y-x$. Then putting $y+x=5^2$, and $y-x=2\times 5$, we get 15^2 as the smallest square required.

In a similar way the least value of N can be found which will

make N^2+aN a square integer, where a is a given constant. For putting the expression equal to y^2

$$N=\{-a+\sqrt{(a^2+4y^2)}\}/2,$$

and the problem is reduced to that of finding the least square of an even number which added to a^2 will make it an integral square.

If a is an even number, N is obviously an integer. If a is odd, $\sqrt{(a^2+4y^2)}$ is also an odd number, and therefore N is again an integer.

Sur un Lieu Géométrique.

Par M. Paul Aubert.

Par un point fixe A *d'une circonférence donnée on mène deux cordes* AB *et* AC *dont le produit a une valeur constante* m^2, *puis on joint* BC. *Trouver* 1° *le lieu du pied* D *de la bissectrice de l'angle* A *du triangle* ABC ; 2° *le lieu des centres des cercles inscrits et ex-inscrits à ce triangle.*

Figure 14.

1°. Soit H le point où la bissectrice AD rencontre la circonférence circonscrite au triangle ABC.

On sait que $AD\times AH=AB\times AC$;

donc on a $AD=m^2/AH$.

Le lieu du point D est la figure inverse de la circonférence lieu de H, le pôle d'inversion étant en A, la puissance d'inversion m^2. C'est donc la perpendiculaire XY au diamètre AO ; elle coupe la circonférence aux points F et G tels que

$$AF=AG=m.$$

2°. Si on mène BK parallèle à XY, les arcs AFB et ACK sont égaux, d'où $\angle\, ACB=\angle\, AB'C'$.

Les droites BC et B'C' sont alors anti-parallèles, et AD est bissectrice de l'angle B'DC. Par suite le cercle inscrit au triangle ABC est aussi inscrit au triangle AB'C', et il en est de même du cercle ex-inscrit dans l'angle A. D'ailleurs on a manifestement

$$AB'=AC,\ AC'=AB,$$

d'où $AB'\times AC'=m^2$.

Le problème revient donc à considérer les triangles tels que

AB′C′, dont la base glisse sur XY de manière que l'on ait toujours la relation $AB' \times AC' = m^2$.

Soient I et I′ les centres du cercle inscrit et du cercle ex-inscrit dans l'angle A à ce triangle.

Le quadrilatère B′IC′I′ est inscrit dans le cercle de diamètre II′, et l'on a

$$AI \times AI' = AB' \times AC' = m^2,$$

car les points B et C sont aussi sur cette circonférence. Si maintenant E, E′ sont les positions de I et I′ qui correspondent au triangle isocèle FAG, on a de même

$$AE \times AE' = m^2$$

d'où

$$AI \times AI' = AE \times AE'.$$

Ainsi les points E, E′, I, I′ sont sur une même circonférence; je dis que cette circonférence est invariable et a pour diamètre EE′. En effet, la polaire du point A par rapport à cette circonférence passe par le point P de XY puisque les points E, E′ sont conjugués harmoniques des points A et P. De même, le faisceau B′(ADII′) étant harmonique, B′D est la polaire de A par rapport à l'angle IB′I′, et D appartient à la polaire de A par rapport à la circonférence EII′E′. Cette polaire est donc la droite XY, et par suite le centre de la circonférence considérée est sur la perpendiculaire AP à cette droite. Son diamètre est donc EE′ et les points I, I′ se déplacent sur cette circonférence invariable qui est le lieu demandé. On voit en passant que AF et AG sont tangentes à cette circonférence aux points F et G, ce qui confirme que le centre est à l'extrémité V du diamètre AO.

3°. Remarquons maintenant, à l'égard des cercles ex-inscrits dans les angles B et C, que si l'on fait tourner le triangle ABC autour de AH de 180°, il viendra s'appliquer sur le triangle AB′C′. Dans ce mouvement la bissectrice extérieure de l'angle A se recouvre elle même, en sorte que si I_2 et I_3 sont les centres des cercles ex-inscrits situés sur cette bissectrice, leurs positions I″ et I‴ après la rotation seront précisément les centres des cercles ex-inscrits au triangle AB′C′.

Or $AI_2 = AI''$ et $AI_3 = AI'''$;

donc le lieu des points I_2 et I_3 est homothétique inverse du lieu des points I″ et I‴ par rapport au centre d'homothétie A, le rapport d'homothétie étant l'unité. Nous sommes donc encore ramenés à considérer les triangles tels que AB'C'.

Cela posé, prolongeons C'A d'une longueur $AB_1' = AB'$, et joignons $I'''B_1'$. On a

$$\angle B_1'I'''I'' = \angle AI'''B' = 180° - \left[\frac{180° - A}{2} + \frac{180° - B'}{2}\right] = \frac{A + B'}{2}.$$

Or $\qquad \angle B_1'C'I'' = (A + B')/2$; donc $\angle B_1'I'''I'' = \angle B_1'C'I''$.

On en déduit $\qquad AI'' \times AI''' = AC' \times AB_1' = m^2.$

Si donc M et M' sont les positions de I'' et I''' qui correspondent au triangle isocèle FAG, on a

$$\overline{AM}^2 = m^2, \text{ ou } AM = AG.$$

Donc I'' et I''' se trouvent sur une circonférence dont le diamètre est la perpendiculaire sur le milieu A de MM'. Je dis que cette circonférence est invariable.

En effet, si L désigne le point d'intersection de I''I''' avec XY, le faisceau B'(LAI''I''') est harmonique. Donc le point L est sur la polaire du point A par rapport à la circonférence en question. Cette polaire devant être perpendiculaire au diamètre qui passe en A est donc XY, et inversement AM est la polaire du point P.

Donc PM est tangente en M à la circonférence considérée, dont le centre est par suite en un point fixe V' défini par la relation

$$\overline{MA}^2 = AP \times AV'.$$

Mais on a déjà $\qquad \overline{MA}^2 = AP \times AV$;

donc $\qquad AV' = AV.$

Ainsi les points I'' et I''' se meuvent sur la circonférence de cercle V' ainsi déterminée.

On voit en passant que A et P sont les points limites de Poncelet relatifs aux cercles qui ont avec V et V' le même axe radical MM'. On déduit de la remarque faite précédemment, que les points I_2 et I_3 se déplacent sur la circonférence VMM' concentrique à la première circonférence lieu des points I et I'. Il y a lieu d'observer que tous les points de ces circonférences ne font pas partie du lieu géométrique, mais seulement certaines portions qu'un examen un peu attentif de la figure permettra de séparer facilement.

Note on the preceding Locus.

By A. J. PRESSLAND, M.A.

The perpendicular from A on BC = AB·AC/2R, which is constant.

Describe a circle with centre A and radius $m^2/2R$. This, which may be called the perpendicular circle, will touch each position of the base, and therefore will touch FG.

Now AD is the axis of the inscribed and the perpendicular circles, and BDC is a common tangent.

Therefore D is their external centre of similitude.

But FG touches the perpendicular circle.

Therefore it touches the inscribed circle.

We have now to find the locus of a point whose distance from a fixed line $= r$ and from a fixed point (the circumcentre) $= \sqrt{R^2 - 2Rr}$, which can easily be proved to be a circle.

The triangle and its escribed parabolas.

By A. J. PRESSLAND, M.A.

In what follows, a triangle ABC is taken, from it another triangle A′B′C′ is formed so that

A is the mid point of B′C′,
B is the mid point of C′A′,
and C is the mid point of A′B′.

From A′B′C′ another triangle A″B″C″ is similarly derived.

If a parabola escribed to ABC touch BC at P, CA at Q, and AB at R, we have shown that AP, BQ and CR intersect in a point O, the locus of which is the minimum ellipse circumscribing ABC.

The equation of PQ is

$$n(x-a) = l(y-b)$$

whence PQ passes through C′.
Similarly QR passes through A′,
and RP passes through B′.

If we join QR and produce it to meet CB in K, and also join PR and produce it to meet CA in L, we find as equation of LK

$$l\{2xb/a - b + y\} = n\{2ya/b - a + x\}$$

an equation which is satisfied by the co-ordinates of the centroid and of the point T where PQ cuts AB.

It is to be noted that O is the pole of LK with respect to the parabola.

The locus of the centroid of PQR will be found to be

$$(3x - 2a)(3y - 2b)\{3ay + 3bx - ab\} = a^2b^2$$

the asymptotes of which are the lines joining the centroids of A′BC, AB′C, and ABC′.

Taking as the equation of the parabola

$$(nx + ly)^2 - 2(nx - ly)(an - lb) + (lb - an)^2 = 0$$

we get for the envelope of the polar of $(x'y')$

$$\begin{aligned} &x^2(y' + b)^2 - 2xy(x'y' - 3ay' - 3bx' + ab) \\ &+ y^2(x' + a)^2 + 2y\,(bx'^2 + 3ax'y' - 3abx' - a^2y') \\ &+ (bx' - ay')^2 + 2x\,(ay'^2 + 3bx'y' - 3aby' - b^2x') = 0 \end{aligned}$$

If we put $x' = a/3, y' = b/3$, we get

$$\frac{x^2}{a^2} + \frac{xy}{ab} + \frac{y^2}{b^2} - \frac{x}{a} - \frac{y}{b} = 0$$

the equation of the minimum ellipse.

Hence the polar of the centroid with respect to the parabola is the tangent at O to the minimum ellipse.

Discussing the envelope equation obtained above we shall find that it represents an ellipse if the point $(x'y')$ be within the triangle A′B′C′, or within any division corresponding to the space between A′C′ produced and B′C′ produced. If the locus be an hyperbola the point $(x'y')$ is in one of the three spaces bounded by A′B′, C′A′ produced, and C′B′ produced, or other similar lines.

If the point $(x'y')$ be on A′B′, or A′C′, or B′C′, the equation from its form represents a parabola, but is really two straight lines which are coincident and pass through one of the points A′,B′,C′.

In other words, A′B′C′ is a self-conjugate triangle with respect to any of the parabolas. This property would seem to point to an easy treatment of the parabola and its self-conjugate triangle, for the median triangle of any self-conjugate can be proved to be circumscribed to the parabola.

If the envelope be a rectangular hyperbola, $(x'y')$ must be on the circle whose equation is

$$\begin{aligned} x^2 + 2xy\cos C + y^2 + 2(c\cos B - 2b\cos C)x \\ + 2(c\cos A - 2a\cos C)y + 2a^2 + 2b^2 - c^2 = 0. \end{aligned}$$

If $(x'y')$ be on the median through A, the envelope passes through A. Hence the envelope is only a circumscribed conic when $(x'y')$ is the centroid.

If $(x'y')$ be on CA, CB or BA, the envelope passes through $(x'y')$.

If $(x'y')$ be A, B, or C, the envelope is two coincident lines, the corresponding medians.

If the envelope be a circle, the point is one of the in-centres or ex-centres of $A''B''C''$.

If $(x'y')$ be the centre of the envelope, it must be either A, B, C, or the centroid.

Fifth Meeting, March 13, 1891.

J. S. Mackay, Esq., M.A., LL.D., ex-President, in the Chair.

On the Use of Dimensional Equations in Physics.

By William Peddie, D.Sc.

Though every quantity, whatever be its nature, has magnitude, no quantity can be said to be large or small *absolutely.* When we speak of the size of any body we mean its size relatively to the size of some other body with which we compare it. A yard is large if we compare it with an inch; it is small when compared with a mile. In the former case the number which represents it is more than 60,000 times larger than the number by which it is represented in the latter case. A mere number is therefore useless as regards the statement of magnitude, except when accompanied by a clear indication of what the thing measured is compared with. The quantity in terms of which the comparison is made is called the *unit*, and the number which tells how often this unit is contained in a given quantity is called the *numeric* of that quantity.

All dynamical quantities may be made to depend upon *three* units only. These are the units of *mass*, of *length*, and of *time*. Thus the unit of speed, being measured by the distance traversed in a certain time, depends upon the unit of length directly, and upon the unit of time inversely. Hence, by doubling the unit of length we double the speed unit, and therefore halve the numeric of any given speed; whereas, by doubling the unit of time, we halve the speed unit, and therefore double the numeric of a given speed. Again, acceleration, being measured by the increase of speed in a certain time, depends upon the unit of speed directly and upon the unit of time inversely; that is, it depends directly upon the unit of length and inversely upon the square of the unit of time. The manner in which the fundamental units are involved in any quantity determines the *dimensions* of that quantity. If M, L, and T, represent the units of mass, length, and time, the dimensions of speed and acceleration are indicated by the symbols $[LT^{-1}]$ and $[LT^{-2}]$ respectively, and the dimensions of energy are $[ML^2T^{-2}]$, and so on.

We may write the expression for Newton's Second Law in the form

$$f = m\frac{l}{t^2},$$

where l is the distance which the mass m moves over from rest, in the time t, under the action of the force f; and it is understood that unit force is that force which, acting upon unit mass, produces unit acceleration. And we may further regard this equation as a dimensional equation; in which case the sign of equality merely means that the dimensions of the quantity on the left-hand side of the equation are identical with those of the expression on the right-hand side of the equation. But, from a dimensional equation, we cannot make any deduction regarding the absolute magnitude of any of the quantities which are involved, for the equation simply asserts *proportionality* of magnitude between its various terms. Still, by a suitable definition of units, we can pass from the dimensional, to the ordinary, equation. Thus, in the above equation, we may define unit force as the force which, acting on unit mass for unit time, causes the unit of mass to move over unit distance from rest; or we might adopt the definition given at the commencement of this paragraph.

The idea of dimensions is of great importance in physics. It affords a useful check on the accuracy of algebraical work; for the dimensions of all the terms in a physical equation must be the same. But its use is not limited to this extent. For example, we may write the equation

$$f = m\frac{l}{t^2},$$

in the form

$$\alpha f = \beta m \frac{\gamma l}{\frac{\beta\gamma t^2}{\alpha}},$$

from which we see that if, *in two similar material systems, the forces, masses, and lengths, are in the ratios* $\alpha/1$, $\beta/1$, $\gamma/1$, *respectively, and if the systems begin to move in precisely similar manners, the motions will continue to be similar provided that we compare them after the lapse of intervals of time which are in the ratio of* $\sqrt{\frac{\beta\gamma}{\alpha}}$ *to unity in the two systems.*

This principle, which was proved (otherwise) by Newton, has been called the *Principle of Dynamical Similarity.* The above proof is due to Bertrand.

Many physical problems can be solved with extreme simplicity by its means. For example, we can determine the speed of propagation of a wave along a stretched cord. Let T be the tension of the cord, and let m be its mass per unit of length. The pressure towards the centre of curvature at any part of the wave T/ρ, where ρ is the radius of curvature and is proportional to the linear dimension l of the wave. Hence the pressure per similar length in each of two such waves is T. And the mass per similar length is ml. Hence the expression for the Second Law of Motion gives

$$\frac{l^2}{t^2} = \frac{T}{m}.$$

But $l/t = v$, the speed of propagation of the wave. Hence the speed is directly as the square root of the tension, and is inversely as the square root of the mass per unit of length.

This result may be directly applied to the problem of the vibrations of a stretched string. For, λ being the length of a wave and the other quantities having the same meanings as above, we get

$$\frac{\lambda}{v} = \frac{\lambda\sqrt{m}}{\sqrt{T}}.$$

Now λ/v is the time of a complete vibration, which is therefore directly proportional to the length of a loop of the string and to the square root of the mass per unit length of the cord, and is inversely proportional to the square root of its tension.

Again, in the case of a gas, of density ρ and under pressure p, the dimensional equation takes the form

$$\frac{pl^2}{l}=\frac{\rho l^2 l}{t^2},$$

from which we conclude that the period of vibration when a disturbance of given wave-length travels through the gas is

$$t=l\sqrt{\frac{\rho}{p}},$$

and that the speed of propagation of that disturbance is

$$v=\sqrt{\frac{p}{\rho}},$$

the proper choice of units being made.

In an oscillatory wave, propagated by gravity, the force is proportioned to the intensity of gravity g, and to the mass of the wave, which is proportional to the density ρ of the liquid and to the square of the length. Hence the fundamental equation takes the form

$$l^2\rho g=\frac{\rho l^2 l}{t^2},$$

from which we get

$$t=\sqrt{\frac{l}{g}}$$

and

$$v=\frac{l}{t}=\sqrt{lg}.$$

This shows that the speed of propagation is proportional conjointly to the square roots of the wave length and of gravity, and is independent of the density of the liquid.

In a ripple, propagated by surface tension, the pressure per unit area of the surface of the ripple is proportional directly to the surface tension T and inversely to the radius of curvature. Hence the pressure per similar area is proportional to T and to the square of the length of the wave. Also the mass is proportional to the density and to the cube of the wave length. Hence we get

$$t = l\sqrt{\frac{\rho l}{\mathrm{T}}},$$

$$v = \frac{l}{t} = \sqrt{\frac{\mathrm{T}}{\rho l}},$$

which shows that the speed of propagation of a ripple is proportional to the square root of the surface tension directly and is inversely proportional to the square roots of the density and the wave length conjointly. Hence we see that, as is well known to be the case, ripples run faster the shorter they are, while gravity waves run faster the longer they are.

As a single additional example of the application of the Second Law, consider the small vibrations of a liquid sphere. Let the radius of the sphere be a, while the density of the liquid is ρ, and its surface tension is T. The force which acts, per unit of area, to restore the spherical form after a slight disturbance is $2\mathrm{T}/a$, and hence the force per similar area is $2\mathrm{T}a$. The mass is proportional to ρa^3, and the ranges of similar disturbances are proportional to a. Hence we get

$$t = a^{\frac{3}{2}}\sqrt{\frac{\rho}{\mathrm{T}}},$$

which shows that the periods of similar vibrations are proportional to the power 3/2 of the radius.

Lastly, as an example of the use of dimensional equations apart from the consideration of the Second Law of Motion, consider the conduction of heat downwards through the earth's crust, for which we have the well-known equation

$$c\frac{dv}{dt} = -\frac{d}{dx}\left(k\frac{dv}{dx}\right),$$

where c is the thermal capacity, k is the thermal conductivity (which we shall assume to be constant), v is temperature, t is time, and x is distance measured downwards from the surface. The quantities dv, dt, dx, being of the same dimensions as v, t, and x, respectively, we get, as a dimensional equation

$$\frac{c}{t} = \frac{k}{x^2}.$$

This gives

$$x = \sqrt{\frac{kt}{c}},$$

which means that if the times are altered in any fixed proportion p (say), the lengths must be altered in proportion of the square root of p in order that the flow of heat may take place under *similar* conditions in the altered circumstances. In other words, *the distances at which similar effects are felt (for example, the distance below the surface at which the periodic variations of surface temperature cease to be felt) are proportional to the square root of the period.*

Again,

$$\frac{x}{t} = \sqrt{\frac{k}{ct}}.$$

Here we may suppose that x represents the length of a wave, while t represents the periodic time, so that the fraction on the right-hand side of the equation is proportional to the rate at which the wave of heat travels downwards. We see therefore that this rate is directly proportional to the square root of the conductivity, and is inversely proportional to the square roots of the thermal capacity and the periodic time conjointly.

It follows from this that, when the period is constant, *the date at which the maximum temperature reaches any given depth is later than the date at which it left the surface in direct proportion to the depth.*

[The law which regulates the diminution of the range of temperature with increase of depth cannot be obtained from the original equation in the way in which we have obtained the two laws just enunciated, for the temperature is not involved in the dimensional equation which we deduce from it.

But we may write the original equation in the form

$$c\frac{dv}{dx}.\frac{dx}{dt} = -\frac{d}{dx}\left(k\frac{dv}{dx}\right);$$

and we may suppose that dx/dt is the speed with which the heat wave travels downwards, in which case the equation becomes

$$\sqrt{\frac{ck}{\mathrm{T}}}.\frac{dv}{dx} = -\frac{d}{dx}\left(k\frac{dv}{dx}\right),$$

where T is the periodic time, and dv/dx now represents the rate at which (say) the maximum temperature changes as the wave passes down.

This equation asserts that the rate of diminution of the rate of change of temperature with depth is proportional to the rate of change itself. In other words, the rate of change diminishes in geometrical progression as the depth increases in arithmetical progression. And its rate of diminution is $\sqrt{c}/\sqrt{k\mathrm{T}}$, k being assumed to be constant. But, since the rate of diminution of the rate of alteration of the range is proportional to the rate of alteration itself, it follows that the rate of alteration bears the same ratio to the range. Hence *the range diminishes in geometrical progression as the depth increases in arithmetical progression*, the rate of diminution being directly as the square root of the thermal capacity, and inversely as the square roots of the conductivity and the periodic time conjointly.]

The above examples will serve to illustrate the extreme ease with which the consideration of dimensional equations leads to the solutions of problems which are usually attacked by the aid of recondite methods alone.

Some relations between the orthic and the median triangles.

By A. J. PRESSLAND, M.A.

FIGURE 15.

Let ABC be the triangle, X, Y, Z the feet of the altitudes, H the orthocentre, A′,B′,C′ the mid points of the sides.

Let ZY meet A′B′ in D, A′C′ in D′, B′C′ in R :
ZX meet B′C′ in E, A′B′ in E′, A′C′ in S :
XY meet C′A′ in F, B′C′ in F′, A′B′ in T.

§ 1. The following triangles are similar to ABC ; AYZ, XBZ, XYC.

B′YD, C′D′Z are similar to AYZ and have parallel sides.
EC′Z, XA′E′ are similar to XBZ and have parallel sides.
XFA′, F′YB′ are similar to XYC and have parallel sides.

Y	is the	internal	centre of	similitude of	the	circles	AYZ, B′YD.
Z	,,	,,	,,	,,	,,	,,	AYZ, C′D′Z.
Z	,,	external	,,	,	,,	,,	XBZ, EC′Z.
X	,,	internal	,,	,,	,,	,,	XBZ, XA′E′.
X	,,	external	,,	,,	,,	,,	XYC, XFA′.
Y	,,	,,	,,	,,	,,	,,	XYC, F′YB′.

Hence the circle AYZ touches the circles B′YD, C′D′Z,
and the circle XBZ „ „ EC′Z, XA′E′,
„ „ XYC „ „ XFA′, F′YB′.

§ 2. U, V, W the mid points of AH, BH, CH are the centres of the circles AYZ, XBZ, XYC. Now A′U is a diameter of the nine point circle; therefore A′Y is a common tangent to the circles AYZ, B′YD, and A′Z is a common tangent to the circles AYZ, C′D′Z.

Hence A′ is the radical centre of the circles
AYZ, B′YD, C′D′Z.
Similarly B′ and C′ are radical centres of triads of circles.

Now AC is the radical axis of B′YD, F′YB′;
and AB is the „ „ „ C′D′Z, EC′Z;
and AB′.AY = AC′.AZ.

Therefore A is the radical centre of B′YD, F′YB′, C′D′Z, EC′Z.
Similarly B and C are radical axes of tetrads of circles.

Hence AA′ is the radical axis of the circles B′YD, C′D′Z; and similarly for BB′ and CC′.

§ 3.
R is the external centre of similitude of the circles B′YD, C′D′Z,
S „ internal „ „ „ „ EC′Z, XA′E′,
T „ external „ „ „ „ XFA′, F′YB′.
It may be shown that
B′C′ is a common tangent to the circles B′YD, C′D′Z.
C′A′ „ „ „ „ „ EC′Z, XA′E′,
A′B′ „ „ „ „ „ XFA′, F′YB′.
Since AA′ bisects B′C′, and A is the radical centre of B′YD, C′D′Z, another proof can be deduced that AA′ is the radical axis of B′YD, C′D′Z.

§ 4. The angle YDB′ = ∠ C.
Therefore the four points A′, Y, D, C are concyclic.
Similarly the following tetrads are concyclic
A′ZD′B; B′EZA; B′XE′C; C′FXB; C′F′YA.
As AY·AC = AZ·AB,

A must be on the radical axis of A′YDC and A′ZD′B. Hence AA′ is the radical axis of these circles.

§ 5. Since $\angle B'C'X = \angle B'YX = \angle B$,
B′C′ touches the circle BXC′.

Therefore the circle C′D′Z touches the circle BXC′F.

Similarly the circle XA′E′ touches the circle A′YDC, and the circle F′YB′ touches AZB′E.

§ 6. Since $\angle ZXB = \angle ZYA' = \angle XCD$,
ZX is parallel to DC.

Similarly BF is parallel to YZ and AE to XY.

If PA′Q be the tangent at A′ to the nine-point circle, and cut XY in P, and AC in Q, then since $\angle B'A'Q' = \angle C = \angle B'DY$, PA′Q is parallel to YZ.

X is the centre of similitude of the quadrilaterals BC′FX, PXA′L. But B,C′,F,X are concyclic.

Therefore P,X,A′,L are concyclic;
and since X is the centre of similitude, the two circles BC′FX and PXA′L touch.

The angle $B'QA' = \angle DYB' = \angle B$,
and $\angle A'XL = \angle XA'L = \angle B$.
Therefore $\angle XLA' = 180° - 2B$.
But $\angle XLA' = \angle XPA'$.
Therefore $\angle XPA' + \angle A'QB' = 180° - B = \angle XA'B'$.

Therefore the circle XA′L touches the circle B′A′Q.

Since B′ is the centre of similitude of B′A′Q and B′YD, the circumcircles of these triangles touch.

Hence from AYZ has been derived the following cycle of circles six in number,
B′YD, AYZ, ZC′D, C′BXF, PXA′L, A′B′Q, B′YD, AYZ,
each of which touches the two adjoining circles.

Other cycles could be obtained from the triangles XBZ, XYC.

As of the four orthic points A,B,C,H any three may be considered as forming the original triangle, it follows that four triangles can be obtained each containing three cycles of six circles.

§ 7. Since ZC′S and DB′C are in perspective,
RS passes through C.
Similarly ST ,, ,, A,
and TR ,, ,, B.

Since ZC′E and DA′C are in perspective,
D′E passes through C.
Similarly F′D ,, ,, B,
and E′F ,, ,, A.

Since ZC is perpendicular to AB, it is bisected perpendicularly by A′B′, and as CD is parallel to ZX, the figure CDZE′ is a rhombus, as are BD′YF and AEXF′.

D′ESR forms a complete quadrilateral two of whose diagonals ZC′ and ZC bisect their corresponding angles and are perpendicular to each other.

Sixth Meeting, April 10, 1891.

J. S. Mackay, Esq., M.A., LL.D., ex-President, in the Chair.

On some properties of a triangle of given shape inscribed in a given triangle.

By R. E. Allardice, M.A.

It is well known that in a given triangle a one-fold infinity of triangles may be inscribed similar to a given triangle This becomes at once obvious on consideration of the converse problem; for we may circumscribe about a given triangle (A), a triangle similar to a second triangle (B), and having its sides parallel to the sides of (B).

We may also show in the following manner that, in a given triangle, one triangle and only one can in general be inscribed having its sides parallel to given directions.

Let D (fig. 16) be a point in the side BC of a triangle ABC; and let DE, EF, FD′, be parallel to the given directions.

Now D and D′ trace out projective ranges on BC; and hence to get the inscribed triangle corresponding to the given directions, we must make D coincide with a double point of this double range. But as D and D′ move in opposite directions along BC and reach infinity together, there is only one double point on this double range.

This way of looking at the problem is of advantage as it suggests a way of extending some of the following theorems to the case of polygons of any number of sides. It should be noted however that in the general case the inscribed polygons will have given angles but will not in general be of given shape.

Another way of looking at the question is to notice that there are only five conditions on the inscribed triangle and that therefore one condition still remains to be satisfied.

Let now (fig. 17), DEF, D′E′F′ be any two triangles of given shape inscribed in ABC; and let corresponding sides of these triangles intersect in the points P, Q, R. We shall show that the triangle PQR is also of constant shape, its angles depending only on those of ABC and DEF.

We have the following relations among the angles in the figures:—

$$\begin{aligned}
&\mathrm{QPR} = \pi - \mathrm{FPQ} - \mathrm{EPR}\,;\\
&\mathrm{FPQ} = \mathrm{BF'D'} = \pi - \mathrm{B} - \mathrm{BD'F'}\,;\\
&\mathrm{EPR} = \mathrm{EE'R} = \pi - \mathrm{C} - \mathrm{CD'E'}.\\
\therefore\ &\mathrm{QPR} = \pi - (\pi - \mathrm{B} - \mathrm{BD'F'}) - (\pi - \mathrm{C} - \mathrm{CD'E'})\\
&\qquad = \mathrm{B} + \mathrm{C} + \mathrm{BD'F'} + \mathrm{CD'E'} - \pi\\
&\qquad = (\pi - \mathrm{A}) + (\pi - \mathrm{D}) - \pi = \pi - (\mathrm{A} + \mathrm{D}).
\end{aligned}$$

Now if we suppose the triangle DEF to remain fixed, while D′E′F′ varies its position, the various positions of PQR will give a series of triangles of given shape inscribed in DEF. If L, M, N, are the points of intersection of corresponding sides of two of the triangles PQR, the triangle LMN will also be of constant shape. In fact, $\mathrm{L} = \pi - (\mathrm{D} + \mathrm{P}) = \pi - \mathrm{D} - (\pi - \mathrm{A} - \mathrm{D}) = \mathrm{A}$.

That is to say the triangle LMN is similar to the triangle ABC with which we started. We obviously get in the same manner an infinity of triangles of constant shape inscribed in any one of the series LMN, and so on *ad infinitum*.

In other words, we get an infinity of similar triangles DEF inscribed in ABC, an infinity of similar triangles PQR inscribed in any one of the set DEF, and so on; and corresponding angles of these triangles recur in the order A, D, P, A, D ……, where the sum of any three consecutive angles is two right angles.

A similar set of propositions may easily be proved regarding triangles of given shape circumscribing a given triangle.

The following is a proof of the known theorem that the triangle DEF is a minimum (has minimum area and therefore also minimum perimeter) when the perpendiculars to the sides of ABC at the points DEF are concurrent; and also of the theorem that if P, Q, R, are the points of intersection of corresponding sides of the minimum triangle and the consecutive triangle, then the perpendiculars at P, Q, R, to the sides of DEF are concurrent.

Let $\angle AEF = x_1$, $\angle AFE = x_2$, $\angle BFD = y_1$, etc.;

then, by drawing perpendiculars from F and E′ on PF′ and PE respectively, we may easily show that

$$EP \cot x_1 = FP \cot x_2$$

$$\therefore \quad PR \sin x_1 \cot x_1 / \sin E = PQ \sin x_2 \cot x_2 / \sin F$$

$$\therefore \quad \frac{\cos x_1}{\cos x_2} \cdot \frac{\cos y_1}{\cos y_2} \cdot \frac{\cos z_1}{\cos z_2} \cdot = \frac{PQ \sin E}{PR \sin F} \cdot \frac{QR \sin F}{QP \sin D} \cdot \frac{RP \sin D}{RQ \sin E} = 1;$$

and hence the perpendiculars at D, E, F, to the sides of ABC are concurrent.

Another proof of this theorem, without the use of infinitesimals, will be given later.

From the fact that the area of DEF is stationary we get,

$$\Delta FPF' + \Delta DQD' + \Delta ERE' - \Delta FQF' - \Delta DRD' - \Delta EPE' = 0,$$

$$\therefore \quad FP^2 + DQ^2 + ER^2 - FQ^2 - DR^2 - EP^2 = 0;$$

and hence the perpendiculars at the points P, Q, R, to the sides of DEF are concurrent. (It may be shown that they concur in the same point).

It may easily be shown that the following construction* gives the minimum inscribed triangle of given shape:—

On the sides of the triangle ABC (fig. 18), described externally, triangles BCA′, B′CA, BCA′, of the required shape. The lines AA′, BB′, CC′ concur in a point Q. From Q draw perpendiculars QD′, QE′, QF′ to the sides of ABC; and from A,B,C, draw perpendiculars to the sides of D′E′F′. These perpendiculars concur in a point P such that AP and AQ, BP and BQ, CP and CQ are isogonal lines. [P and Q are frequently called inverse points; but as I shall have occasion to speak of points inverse with reference to a circle, I shall call points such as P and Q reciprocal points. This name is appropriate since the trilinear co-ordinates of Q are the reciprocals of the co-ordinates of P.]

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* See a paper by Mr M. Jenkins in the Quarterly Journal of Mathematics, vol. xxi. (1886), p. 84.

If perpendiculars PD, PE, PF be drawn from P to the sides BC, CA, AB, the triangle DEF will be similar to A'BC, the angles corresponding in this order.

It may easily be shown further that the angles A, D and D' make up two right angles so that D'E'F' is similar to the triangle PQR of fig. 17.

By means of the converse of the theorem that if a parabola touches the three sides of a triangle the circumcircle of the triangle passes through the focus, it may easily be proved that the envelopes of the sides of the inscribed triangle of a given shape are three parabolas. (Compare the paper by Mr Jenkins already referred to). For let P (fig. 19) be the point whose pedal triangle has the given shape; then, if PD, PE, PF be drawn so that PEAF, PFBD, PDCE are cyclic quadrilaterals, it may easily be shown that the triangle DEF is of invariable shape. Hence the sides of DEF envelope three parabolas of which P is the common focus.

We can now get the condition that the triangle DEF (fig. 19) be a minimum very simply by means of the propositions just proved. For if R be the (variable) radius of the circle circumscribing AEPF, we have $EF = 2R\sin A$; and therefore, as A is constant, EF is a minimum when R is a minimum, that is, when the circle is described on AP as diameter. Moreover, in this case, EF is the tangent to the parabola at the vertex; and therefore the foot of the perpendicular from P on FE is the vertex of the parabola, and is the point of intersection of two consecutive positions of EF, when EF is a minimum, as was proved above.

It may be noted that the above propositions, taken together, give the following property of the parabola :—

Let any three confocal parabolas be described so that each touches two sides of a triangle ABC. Take any point D in BC and draw a tangent to the first parabola meeting CA in E; draw a tangent from E to the second parabola meeting AB in F; then the tangent from F to the third parabola will pass through D, and the triangle DEF will be of invariable shape whatever be the position of D. Further, if R, P, Q, be the points of contact of these tangents, the triangle PQR will also be of invariable shape, and the angles will satisfy the relations $A + D + P = B + E + Q = C + F + R =$ two right angles.

A construction was given above for the minimum inscribed triangle of a given shape and for what may be called the point corresponding to the series of triangles of a given shape.

In the course of the construction it came out that, in order to make the problem definite, it must be stated in which of the sides of ABC the different vertices of DEF are to lie. By varying the positions of these vertices in every possible way we get six series of triangles of an assigned shape, with, of course, a definite point corresponding to each series. Further, in that construction, the triangles were described *externally* on the sides of ABC. But if triangles be described *internally* on the sides of ABC another series of similar triangles is obtained with a corresponding point. The relation between the series of inscribed triangles obtained by describing triangles (1) externally and (2) internally on the sides of ABC is that any triangle of the one series is inversely similar to any triangle of the second series. This is a point of some interest as it shows that as regards the problem in hand, two triangles that are inversely similar are perfectly distinct and have to be considered separately. Of two such triangles one cannot of course be transformed into the other by continuous variation.

We thus get twelve series of triangles of an assigned shape with a point corresponding to each series; and the question naturally arises, What is the relation between these twelve points?* In order to answer this question it is necessary to consider some of the properties of the three circles, each of which has for diameter the line joining the points dividing one of the sides internally and externally in the ratio of the other two. [These circles are sometimes known as the circles of Apollonius; and it will be convenient to call them by that name, to avoid circumlocution]. The following are some of the properties of these circles :—

(1) They have two points in common; namely the points the distances of which from the vertices are inversely proportional to the opposite sides. They have therefore a common radical axis.

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* A useful hint in this connection was supplied by the following question in the *Educational Times* :—10695 (Professor Neuberg.)—Soient P,P′ deux points inverses par rapport à un cercle d'Apollonius du triangle ABC. Démontrer que les triangles podaires de ces points par rapport au triangle ABC sont inversement semblable.

(2) They cut the circumcircle orthogonally; and hence the circumcentre lies on their common radical axis.

This may be proved as follows:—

Let DD′ (fig. 20) be the diameter of one of the circles of Apollonius of the triangle ABC, and let O be the circumcentre.

$$\angle OAB = C - \Pi/2, \quad \angle DAP = \tfrac{1}{2}A + B;$$
$$\therefore \angle OAP = (C - \Pi/2) + (\tfrac{1}{2}A + B) + \tfrac{1}{2}A = \Pi/2.$$

It follows from this that the points of intersection of the circles of Apollonius are inverse with respect to the circumcircle.

(3) One of the circles cuts each of the other two circles at an angle of 60°; and the two latter therefore intersect at an angle of 120° (v. *Lady's and Gentleman's Diary*, 1845, p. 59).

It may be noted that the central axis and the radical axis of the circles of Apollonius are two straight lines at right angles to one another which are symmetrically related to the triangle.

Returning to the question of accounting for the twelve points the pedal triangles of which are similar, we shall begin by proving the following proposition which has been already referred to:—

The pedal triangles of two points that are inverse to one another with respect to one of the circles of Apollonius are inversely similar.

Let P and P′ (fig. 21) be inverse points with reference to the circle of Apollonius passing through A, whose centre is L; and let DEF, D′E′F′, be the pedal triangles of P and P′, D and D′ lying in BC, etc. It may easily be seen from fig. 19 that $\angle D = PBA \pm PCA$ (the sign being taken according as the two angles, read in this way, are traced out in opposite directions or in the same direction).

In the figure we have

$$\begin{aligned} PCB &= \pi - PP'B = AP'P + P'BA + P'AB \\ &= PAL + P'BA + P'AB \\ &= PAC + CBA + P'BA + P'AB \\ \therefore \quad PCB + PAB &= (CBA + P'BA) + (PAC + P'AB + PAB) \\ &= P'BC + P'AC; \end{aligned}$$

that is $E = F'$.

Similarly we may show that $F = E'$, $D = D'$.

Hence when we take points inverse with respect to the circle passing through A, the vertices lying on BC are corresponding vertices, but those lying on CA and AB are not.

If therefore we start with any point, take its inverse with reference to each of the circles of Apollonius, take the inverses of these points again, and so on as long as we get any new points, the pedal triangles of all the points we get in this way must be similar. An upper limit to the number of points that may be so obtained, is given by the fact proved above that there are only twelve such triangles; and it might be shown *à priori* without much difficulty that more than six points cannot be obtained in this way. In fact it is obvious that the two points whose pedal triangles are inversely similar, but have corresponding vertices on the same sides of the triangle ABC, cannot be derived from one another by this method.

We shall now show that there are in general six points and only six that may be derived from one another by repeated inversion with reference to the circles of Apollonius.

Given two circles (fig. 22) with centres A and B; let B′ be the inverse of the point B with reference to the circle A and A′ be the inverse of the point A with reference to the circle B. To find the condition that the inverses of B′ with reference to the circle B and of A′ with reference to the circle A coincide (in a point T, say).

Let the circles intersect in P.

Since B and B′ are inverse points with reference to A, $APB' = PBA$; similarly $BPA' = PAB$, and $TPB = PB'B = PAB' + APB'$.

$\therefore$ $A'PT = B'PA$ and $B'PT = A'PB$.

Hence $APT = BPT = PAB + PBA = 60°$; and $APB = 120°$.

In fig. 23 the point T does not lie between A and B and APB is 60°.

The condition of this theorem is satisfied by the circles of Apollonius, as these intersect at angles of 60° and 120°. In this case the point T is the centre of the third circle.

Take now (fig. 24) two circles intersecting at 120° (or at 60°); and let P be any point.

Let Q be the inverse of P with reference to the circle A,
,, R ,, ,, ,, P ,, ,, ,, ,, B
,, T ,, ,, ,, Q ,, ,, ,, ,, B,
,, S ,, ,, ,, R ,, ,, ,, ,, A

Let AT and BS intersect in O; to show that O is the inverse of T with reference to the circle A and also of S with reference to the circle B.

The points PQSR and the points PQTR are concyclic; and hence the five points PQSTR are concyclic.

Let now B′ be the inverse of B with reference to the circle A
„ „ A′ „ „ „ A „ „ „ „ B
„ „ U „ „ „ B′ „ „ „ „ B
and also U „ „ „ A′ „ „ „ „ A.

The four points BB′QP are concyclic. Invert the circle on which these points lie, with reference to the circle B. The straight line which is the inverse of this circle passes through U, T and R; and hence the straight line RT passes through U. Similarly QS passes through U.

Hence by the properties of Pascal's Hexagram O must lie on the circle passing through P,Q,S,T,R.

No new point can be got by inverting any of these six points with reference to any of the circles of Apollonius.

Hence we see that six of the twelve points the pedal triangles of which are similar, are got by inverting any one of the points, in every possible way, with reference to the circles of Apollonius, and that the six points obtained in this way all lie on the circumference of a circle which cuts the circles of Apollonius orthogonally.

We shall next proceed to account for the other six points and shall begin by proving the following theorem:—

If P and P′ (fig. 25) are inverse points with reference to the circumcircle of a triangle ABC, then the pedal triangles of P and P′ are inversely similar and have corresponding vertices on the same sides of ABC.

Let O be the circumcentre; DEF, D′E′F′, the pedal triangles of P and P′.

Then

$$\angle E = \angle PAB + \angle PCB = \angle PAO + \angle PCO + \angle B = \angle AP'C + \angle B;$$

and

$$\angle E' = \angle P'AF' + \angle P'CD' = \angle AP'C + \angle B = \angle E.$$

Hence the remaining six of the twelve points under consideration are got by inverting the six already obtained, with reference to the circumcircle. The new set of six points will of course also lie on a circle cutting the circles of Apollonius orthogonally; and the two circles on which the two sets of six points lie are inverse to one another with reference to the circumcircle. It may be noted that all the six points belonging to one set are obtained, according to the construction given above, by constructing triangles of the required shape on the sides of ABC *externally;* while those belonging to the other set are obtained by constructing the triangles on the sides of ABC *internally.*

It may be of interest now to consider the positions of these twelve points in certain particular cases.

In the first instance, let the pedal triangle of any one of the twelve points be similar to the given triangle. In the set obtained by constructing the triangles externally one of the points is the circumcentre and two others are the Brocard points; and hence the circle on which the six points lie is the Brocard circle. The other three points lie on the symmedians and are therefore the points where the symmedians meet the Brocard circle. [It is known that the symmedian point lies on the Brocard circle; but its pedal triangle is not similar to ABC.] As a consequence of previous theorems, these six points are all inverses of one another with reference to the circles of Apollonius; and the Brocard circle cuts the circle of Apollonius orthogonally.

Considering now the other set of six points we may easily show that three of them are the centres of the circles of Apollonius and hence the circle on which these six points lie becomes in this case a straight line, namely the central axis of the circles of Apollonius. [This may easily be seen otherwise, as follows. Since the circle on which the six points lie is the inverse of the Brocard circle with reference to the circumcircle, the centre of which lies on the Brocard circle, it must be a straight line; and the only straight line that cuts the circles of Apollonius orthogonally is their central axis.]

One of the three remaining points is at infinity, being the inverse of the circumcentre; and the other two are the inverses of the centres of two of the circles of Apollonius with reference to the third circle. These two points are the inverses of the Brocard points with reference to the circumcircle. It may easily be shown that the central axis of the circles of Apollonius is the radical axis of the circumcircle and the Brocard circle.

A few other results may be stated without proof. The demonstrations will be found to be in all cases very simple.

Let P be any point in the plane and DEF its pedal triangle, D lying in BC, E in CA and F in AB. We may impose any conditions (as regards shape) on DEF and look for the corresponding locus of P.

If the ratio DE : DF is constant $(=l:m)$ the locus of P is a circle on QR as diameter where Q and R are points dividing BC internally and externally in the ratio $mc : lb$. It should be noted that the twelve points which have similar pedal triangles will not all lie on this circle.

If DEF is isosceles (with DE = DF) the locus of P is the circle of Apollonius passing through A. Since all the points whose pedal triangles are isosceles lie in one or other of the circles of Apollonius, it is evident that all the six points that give isosceles triangles of a given shape lie on the circles of Apollonius, two on each. [The twelve points of the general case reduce to six when the pedal triangle is isosceles.] From this it may easily be seen that of the three circles of Apollonius any two are inverse to one another with reference to the third. If further any point on one of the circles of Apollonius be inverted in every possible way with reference to these circles, only three points in all can be obtained; and these together with their inverses with reference to the circumcircle are the six points that have pedal triangles isosceles and of given shape.

As a particular case of the above, the points whose pedal triangles are equiangular must lie on all the circles of Apollonius and must be one or other of the two points common to these three circles. This is a well-known property of these two points.

If in the triangle DEF the angle D is constant, the locus of P will be a circle passing through B and C. If we make D equal to two right angles, the angles E and F will also be constant, each being zero, and the locus of P will be the circumcircle. Hence we get the property of the pedal line (or so-called Simson line) as a particular case of the above theory. A flat triangle is the only triangle the angles of which may be assigned without determining the shape of the triangle. The general method given above for determining the twelve points whose pedal triangles have a given shape may be applied even in the case where the triangles are flat ones. The twelve points reduce in this case to six.

A particular case of this again that may be specially considered is when the flat triangle is isosceles. There are three points on the circumcircle the pedal lines of which may be considered as flat isosceles triangles. The above method applied to the construction of these points shows that they are the points where the symmedians meet the circumcircle. Since their pedal triangles are isosceles these points must lie on the circles of Apollonius; hence we get the theorem that the symmedians are the common chords of the circumcircle and the circles of Apollonius.

Seventh Meeting, May 8, 1891.

R. E. Allardice, Esq., M.A., F.R.S.E., President, in the Chair.

On the solitary wave.

By John M'Cowan, M.A., B.Sc.

[This paper is printed in the *Philosophical Magazine*, July 1891.]

On some applications of the pedal line of a triangle.

By Professor A. H. Anglin.

1. Taking the two following known properties of the pedal line of a triangle, viz. :

I. *The locus of a point, such that the feet of the perpendiculars from it on the sides of a triangle are collinear, is the circum-circle of the triangle ;*

II. *The pedal line bisects the distance between the orthocentre and the corresponding point in the circumference of the circum-circle ;*

we may apply them to establish the following known theorems :—

(1.) *The circum-circle of the triangle formed by three tangents to a parabola passes through the focus.*

For, the feet of the perpendiculars from the focus on the tangents lie on a straight line, viz., the tangent at the vertex ; hence by (I.) the focus is on the circumference of the circum-circle.

(2.) *The orthocentre of the triangle is on the directrix.*

For, if O be the orthocentre, by (II.) SO is bisected by the pedal line corresponding to S, that is, by the tangent at the vertex ; hence, if OX be perpendicular to the axis, SA = AX, and therefore OX is the directrix.

2. We may also, by reference to the pedal line, *find the equation to the circum-circle of a triangle formed by three lines the equations to which are given in Cartesian co-ordinates.*

We have to express analytically the fact that the area of the triangle formed by joining the feet of perpendiculars on the sides of the given triangle from the point (x, y) is zero, since it becomes a pedal line of the given triangle.

If, therefore, DEF be the pedal line of the triangle ABC corresponding to the point P, we have

$$\Delta PFE + \Delta PDE - \Delta PFD = 0,$$

that is,

$$PF \cdot PE \sin A + PD \cdot PE \sin C - PF \cdot PD \sin B = 0\,;$$

so that, if the equations to the sides of the triangle ABC be of the form $x\cos\alpha + y\sin\alpha - p = 0$, the equation to the circum-circle is

$$(x\cos\alpha_2 + y\sin\alpha_2 - p_2)(x\cos\alpha_3 + y\sin\alpha_3 - p_3)\sin(\alpha_2 - \alpha_3)$$
$$+ \text{two similar terms} = 0.$$

If the equations to the sides of the triangle be given in the form $ax + by + c = 0$, the equation to the circum-circle is

$$(a_2b_3 - a_3b_2)(a_1^2 + b_1^2)(a_2x + b_2y + c_2)(a_3x + b_3y + c_3)$$
$$+ \text{two similar terms} = 0.$$

[We may also show absolutely that this equation does represent the circum-circle. For the locus represented by it goes through A, B, C, since the co-ordinates of each obviously satisfy the equation; and, further, the equation represents a circle, since on examination it will be found that the coefficients of x^2 and y^2 are equal, while that of xy is zero.

The pedal line of a triangle and theorems connected with it.

By George A. Gibson, M.A

Most of the following theorems occur in a more or less explicit form in text-books on the geometry of the parabola; but it will not, I hope, be without interest and value to consider them independently, and to prove them by using only the propositions of Euclid.

1. If the perpendiculars AD, BE, CF of a triangle ABC are produced to meet the circumcircle of the triangle in X, Y, Z respec-

tively; and if through the orthocentre O any line be drawn meeting the sides BC, CA, AB in U, V, W respectively, then XU, YV, ZW intersect the circumcircle at the same point S, the pedal of which is parallel to the line through O and is midway between S and that line.

From the orthocentre O (fig. 26) draw any line OU cutting BC at U and let XU meet the circumcircle at S.

Draw SL perpendicular to BC, meeting OU at P.

Assume the property that OD is equal to DX.

Then since the triangle ODU, XDU are congruent, so are the triangles SLU, PLU and therefore SL = LP.

Also $\angle$ UOX = $\angle$ UXO = $\angle$ SCA.

Similarly it may be shown that if SY meet CA in V and if SM be drawn perpendicular to CA and produced to meet OV at Q, then SM = MQ and $\angle$ VOY = $\angle$ SCB.

Hence $\angle$ UOX + $\angle$ VOY = $\angle$ ACB = supplement of $\angle$ XOY.

Therefore OU and OV are in the same straight line.

Similarly it may be shown that ZS meets AB at W, the point of intersection of UO and AB, and that if SN be drawn perpendicular to AB and produced to meet UW at R, then SN = NR.

L, M, N being the middle points of SP, SQ, SR respectively, are therefore collinear, and obviously LM bisects SO.

2. If on the pedal line a point G be taken (fig. 27) and if through G a line be drawn perpendicular to SG meeting the sides BC, CA, AB in A′, B′, C′ respectively, then SA′, SB′, SC′ make equal angles with the sides BC, CA, AB.

The points S, G, L, A′ are concyclic, since the angles SGA′, SLA′ are right angles. Hence the angle SA′L is equal to the angle SGN. In the same way it may be shown that the angles SB′M, SC′N are each equal to the angle SGN and therefore to the angle SA′L.

It is easy to show, conversely, that if A′, B′, C′ are three collinear points on the sides such that the angles SA′B, SB′A, SC′A are equal, then S lies on the circumcircle of the triangle and the pedal line of S meets A′B′ at G so that SG is perpendicular to A′B′.

3. If in fig. 27, A′B′C′ be any line meeting the sides of the triangle in A′, B′, C′, then it is readily shown that the circumcircles of the four triangles formed by the four lines, AB, BC, CA, A′B′

intersect in some point S. All four triangles have obviously the same pedal line with respect to S. Hence, if O be the orthocentre of the triangle ABC, it follows from (1) that the orthocentres of all four triangles lie on the line through O parallel to the pedal line of S.

It is also readily seen that the lines joining S to the points in which any one of the four lines cuts the other three are equally inclined to these three.

4. In fig. 26 it is not necessary that O should be the orthocentre in order that XU, YV, ZW should intersect the circumcircle of the triangle at the same point; all that is necessary is that AX, BY, CZ should be concurrent at the point O through which the line UVW is drawn.

The converse is also true, viz., if any line cut the sides at U, V, W and if any point S on circumcircle be joined to U, V, W and produced to meet the circumference at X, Y, Z, then AX, BY, CZ will be concurrent at some point O on the line UVW.

The proof is, in both cases, given immediately by Pascal's Theorem. Thus if, as in (1), we suppose OU drawn, XU produced to S and SY joined cutting CA in V, then applying Pascal's Theorem to the hexagon AXSYBC we see that AX, YB intersect at O, XS, BC at U, SY, CA at V and therefore V lies in UO produced. Similarly it may be shown that W lies in UO produced.

It may be noticed that if the line be the pedal line of the point in the converse theorem, then the point of concurrence is the point at infinity on the line. Thus if in fig. 26, SL meet the circle at x, then Ax is parallel to LM, for

$$\angle SxA = \angle SBA = \angle SLN \text{ since SBLN is a cyclic quadrilateral.}$$

The figure $AxPO$ is therefore a parallelogram; xP is equal to AO and the mid point of AP, being also the mid point of Ox, is on the nine-point circle of the triangle.

5. If A′C′B′ be any line drawn according to the construction of paragraph (2), then the ratio of the intercepts A′C′, C′B′ is constant, and also the ratios BC′:CB′, CA′:AC′, AB′:BA′: for (fig. 27)

the triangles A′C′S, LNS are similar

$$\therefore\ A'C' : C'S = LN : NS.$$

In the same way from the triangles SC′B′, SNM

$$C'S:C'B' = NS:NM$$

$$\therefore\ A'C':C'B' = LN:NM \text{ a constant ratio.}$$

Again, the triangles BC′S, CB′S are equiangular, for

$$\angle SBC' = \angle SCB' \text{ and } \angle SC'B = \angle SB'C$$

$$\therefore\ BC':CB' = BS:CS.$$

Similarly CA′:AC′ = CS:AS, AB′:BA′ = AS:BS.

Further if any other point H were taken on LM and a line drawn through H perpendicular to SH and cutting BC, CA, AB, B′C′ in T, U, V, W, then if the first four lines be fixed while TW varies, the ratios UV:VW and UT:WT are constant so that the cross ratio of the points T, U, V, W is constant and the ratio of the three intercepts UV, VW, WT is constant.

6. It is obvious that the properties deduced in paragraph 5 are those of tangents to a parabola with focus S. It might be easily shown, for example, that when A′B′ is consecutive to BC, the point A′ is such that A′B:BC is equal to LN:NM and $\angle SA'L = \angle SLN$. Many of the properties of the parabola when it is determined by means of its tangents are very easily deduced by constructing the triangle formed by three of them and proceeding as in 5.

Thus, given four tangents, draw the tangents which pass through a given point.

The second point, S, in which the circumcircles of any two triangles formed by the four tangents intersect is the focus and the pedal line of S is the tangent at the vertex. If on the line joining S to the given point P a circle be described cutting the pedal line in Q and R, PQ and PR are the tangents.

Or again, if on the sides AB, AC of a triangle (fig. 27) two points B′, C′ be taken such that BC′:CB′ = constant, then the envelope of B′C′ is a parabola inscribed in the triangle and the locus of any point I′ dividing B′C′ in a given ratio is the (second) tangent through I where BI:IC = given ratio.

7. As another application to the parabola, I had selected the theorem that the centre of every triangle self-polar with respect to a parabola lies on the directrix. The proof required the theorem that the circles on the three diagonals of a complete quadrilateral have a common radical axis, namely the line containing the four

orthocentres of the four triangles determined by the four sides of the quadrilateral. I reached a proof that seemed to leave nothing to be desired in point of simplicity; but I have since found that it is practically the same as that given in *Townsend's Mod. Geom.*, vol. i., p, 253, and I have therefore struck it out.

It may be noted, however, that this elementary demonstration gives at once the theorem on the self-conjugate triangle, for the circle circumscribing the diagonal triangle cuts the circles on the three diagonals orthogonally and hence its centre must lie on the radical axis of the circles, *i.e.*, on the directrix of the parabola which is inscribed in the quadrilateral. Further the line of collinearity of the mid-points of the three diagonals must be parallel to the axis of the parabola.

8. Several of the ordinary standard theorems on the parabola are readily deducible as special cases of the properties of the complete quadrilateral mentioned in the preceding paragraph.

(i) Let, in fig. 27, AB, A'B' become consecutive to CA, BC respectively and suppose their points of contact *b*, *a*. Then the diameter of the quadrilateral becomes the line through C and the mid-point of the chord *ab* and by the theorem of (7) this line is parallel to the axis of the parabola.

(ii) Again, let AB remain while A'B' becomes consecutive to BC, then the diameter of the quadrilateral becomes the line through the mid point of BC and the mid point of A*a*. Hence if any triangle circumscribe a parabola, the mid point of any side and the mid point of the line from the point of contact of that side to the opposite vertex are on a line parallel to the axis of the parabola.

(iii) Suppose now the tangent AB (fig. 27) bisected at its point of contact *c*, then by (ii) *c* and the mid point of C*c* are on a line parallel to the axis, *i.e.* C*c* is parallel to the axis. Hence by (i) C, *c*, and the mid point of the chord of contact *ab* are on a line parallel to the axis. In other words, any point *c* on the parabola, the mid point of any chord *parallel* to the tangent at *c* and the point of intersection of the tangents at the ends of the chord are on a line parallel to the axis—the fundamental property of a diameter. By the principle of paragraph (5), A, B are the mid points of *cb*, *ca* when *c* is the point of AB and therefore C*c* = *cd* if *d* be the mid-point of *ab*.

(iv) When AB, A'B' become consecutive to CA, BC as (i), the circles on BB′, CC′ as diameters become the point circle C, and the directrix is by (7) the radical axis for the circle on ab as diameter and the point circle C. Hence if C be on the directrix the circle on the chord ab must touch it at C and therefore the tangents from a point on the directrix are at right angles.

(v) Again taking d as the mid point of the chord ab and assuming that ad^2 varies as Cd, where C is the point of intersection of tangents at ends of a, b, then the distance of d from the image of C in the directrix will be constant for all tangents drawn from a point on the same diameter.

For the image of C, say e, is the second limiting point for the system C and circle on ab. Hence $de.d\mathrm{C} = ab^2$; ∴ de is constant.

Eighth Meeting, June 12, 1891.

R. E. Allardice, Esq., M.A., F.R.S.E., President, in the Chair.

On a Representation of Elliptic Integrals by Curvilinear Arcs.

By John M'Cowan, M.A., B.Sc.

It is well known that the elliptic integral of the second kind may be represented by the arc of an ellipse, and mathematicians have sought with various success to represent similarly by the arc of an algebraic curve the elliptic integral of the first kind. The general solution of the problem has not been obtained, but Serret and Cayley have given solutions of a very general character.

If, however, the condition that the curve be algebraic be not imposed solutions may be obtained without difficulty. That here given has the peculiarity that the set of elliptic integrals of the first kind for all values of the modulus is represented by the arcs of one system of curves, in several distinct ways, each of a very simple geometric character. The system of curves is further interesting, as representing in the different groupings which naturally arise different systems of stream lines due to vortices in two dimensional fluid motion.

These peculiarities have induced me to make this communication.

§ 1. *To find a curve, the arc of which will represent the integral*

$$s = \int_0^\psi \frac{a d\psi}{\sqrt{1 - k^2\sin^2\psi}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Let the curve be referred to rectangular co-ordinates x and y; let s be the length of the arc measured from the point of intersection of the curve and the axis of y to the point x, y, and let ψ be the inclination of the tangent at this point to the axis of x.

Now $$\frac{dx}{ds} = \cos\psi, \; \frac{dy}{ds} = \sin\psi$$

$$\therefore \quad x = \int_0^\psi \cos\psi ds = \int_0^\psi \frac{a\cos\psi d\psi}{\sqrt{1 - k^2\sin^2\psi}} \quad \ldots \quad \ldots \quad (2)$$

$$\therefore \quad kx = a\sin^{-1}k\sin\psi. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$y = \int_0^\psi \sin\psi ds = \int_0^\psi \frac{a\sin\psi d\psi}{\sqrt{1 - k^2\sin^2\psi}} \quad \ldots \quad \ldots \quad (4)$$

$$\therefore \quad ky = -a\log(k\cos\psi + \sqrt{1 - k^2\sin^2\psi})/(1+k). \quad \ldots \quad (5)$$

The equation to the curve, found by eliminating ψ between (3) and (5) is therefore

$$\cosh ky/a - \cos kx/a = k\sinh ky/a. \quad \ldots \quad \ldots \quad (6)$$

It may be noted that (6) holds for all real or pure imaginary values of k, and therefore for all such values this system of curves, all passing through the origin, represents the integral (1), s being measured from the origin. We proceed to show, however, that the system represents elliptic integrals of the first kind in other ways, equally simple.

§ 2. *Two other interpretations.*

For convenience put $k/a = m$, $\eta = 1/k$, and change the independent variable from ψ to x in (1) by means of (3). We thus obtain

$$s = \int_0^x \frac{dx}{\sqrt{1 - \eta^2\sin^2 mx}} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Hence the system (6) represents this integral also, but in an entirely different manner, the amplitude being now the abscissa, instead of the inclination of the tangent.

There is still another simple interpretation.

Taking the common form of quadric transformation

$$2\phi = \psi + \sin^{-1}k\sin\psi = \psi + mx, \text{ by } (3)$$

(1) becomes

$$s = \frac{2a}{1+k}\int_0^{\frac{1}{2}(\psi+mx)} d\phi / \sqrt{1 - c^2\sin^2\phi} \qquad \cdots \qquad \cdots \quad (8)$$

where $c^2 = 4k/(1+k)^2$. The transformation applied to (7) leads to the same result.

Thus in this third simple geometric manner the elliptic integrals of the first kind are represented by the system of curves (6).

§ 3. *Connexion with stream lines.*

Remembering that $\eta = 1/k$, we may write (6),

$$\eta = \frac{\sinh my}{\cosh my - \cos mx} \qquad \cdots \qquad \cdots \qquad \cdots \quad (9)$$

whence,

$$\frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2} = 0\,.$$

Thus η may be regarded as the stream function in a case of two dimensional fluid motion. Further, η is everywhere finite except at the points on the plane $y = 0$, where $\cos mx = 1$, hence where y is small, and $mx = 2n\pi + mx'$ where x' is small, (9) becomes

$$\eta = \frac{my}{m^2(x'^2 + y^2)} \qquad \cdots \qquad \cdots \qquad \cdots \quad (10)$$

Hence, η is the stream function due to a set of equal straight and parallel vortex filaments adjacent to an infinite plane wall in infinite liquid, at equal distances $2\pi/m$ apart.

The stream lines $\eta = \text{constant}$, are thus easily pictured. When $\eta > 1$, or $k < 1$, they are closed curves, nearly circular by (10) when k is small, touching the axes of x at the vortices and always lying within the limits $x' = \pm\frac{\pi}{2m}$ on either side of each vortex. The curve for which $k = 1$ is asymtotic to the planes $x' = \pm\frac{\pi}{2m}$ on either side of each vortex, and separates the closed curves for which $k < 1$ from the curves for which k is > 1, which pass from vortex to vortex touching the axis of x at each.

§ 4. *Second representation by stream lines.*

The system of curves given by (6) may remarkably enough be grouped in other ways, giving other stream line systems. If in the integration (4) we take a different inferior limit, we may take instead of (4)

$$y = -\frac{a}{k}\sinh^{-1}\frac{k}{\sqrt{1-k^2}}\cos\psi,\ k<1.... \qquad ... (11)$$

or

$$y - \frac{a}{k}\cosh^{-1}\frac{k}{\sqrt{k^2-1}}\cos\psi,\ k>1.... \qquad ... (12)$$

Hence, by (2), the system of curves is given by

$$\cos mx = \sqrt{1-k^2}\cosh my,\ k<1. \quad .. \qquad ... (13)$$

or

$$\cos mx = \sqrt{k^2-1}\sinh my,\ k>1. \quad ... \qquad ... (14)$$

The curves given by (13) and (14) are of course the same as those given by (6), but they are differently situated relatively to one another.

The symmetry of the curves is shown very obviously by these latter equations.

If in (13) we put $k = \operatorname{sech}\xi$, it may be written

$$\xi = \tanh^{-1}\frac{\cos mx}{\cosh my} \quad ... \quad ... \quad ... (15)$$

and hence we may show, much as was done for η, that ξ is the stream function for an infinite straight vortex filament lying midway (at the origin) between two infinite plane walls, $x = \pm\frac{\pi}{m}$.

The curves (14) would cut (13) orthogonally if they were displaced through a distance $\pi/2m$ along the axis of x, and are therefore the equipotential surfaces for the same vortex shifted through this distance. In fact, putting $k = \operatorname{cosec}\zeta$ in (14), it becomes

$$\zeta = \tan^{-1}\frac{\sinh my}{\cos mx} \quad ... \quad ... \quad ... (16)$$

and if we shift the origin through a distance $x = \pi/2m$ this becomes the velocity potential corresponding to the stream function ξ.

§ 5. The method applied to the elliptic integral of the first kind in equations (1) to (6) is obviously equally applicable to the other elliptic integrals and to a very large class of integrals depending on them, to give curves which will represent them by their arcs. The simple threefold geometric interpretation which has been given for the integral of the first kind is, however, peculiar to that integral, and in general the stream line connexion is so also.

On the transformation and classification of Permutations.

By T. B. Sprague, M.A.

Let $(a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots a_n)$ represent any permutation of the first n natural numbers. If we take the first number, a_1, and put it last, we get a new permutation; and if we perform this operation $(n-1)$ times, we get the $(n-1)$ permutations

$$\begin{array}{l}(a_2, a_3, \ldots\ldots\ldots\ldots\ldots\ldots a_n, a_1)\\ (a_3, a_4, \ldots\ldots\ldots\ldots\ldots a_n, a_1, a_2)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ (a_n, a_1, a_2 \ldots\ldots\ldots\ldots\ldots\ldots a_{n-1}).\end{array}$$

These, together with the original permutation, make n in all. All these n permutations are evidently different from each other, but they all belong to the same cyclical series, $a_1, a_2, a_3 \ldots\ldots\ldots a_n, a_1, a_2, \ldots$, which for brevity we will call a "cycle". It is clearly immaterial with what number in a cycle we begin, but it will be convenient to begin always with 1. Let the process of deriving one of our permutations from the preceding one, be denoted by t, so that

$$t(a_m, a_{m+1}, \ldots\ldots\ldots a_{m-1}) = (a_{m+1} \ldots\ldots\ldots a_{m-1}, a_m);$$

then, if we put P for our original permutation, the n permutations will be denoted by

$$\mathrm{P},\ t\mathrm{P},\ t^2\mathrm{P}, \ldots\ldots\ldots t^{n-1}\mathrm{P}.$$

It is obvious that, by repeating the process n times, we get the original permutation over again, so that $t^n\mathrm{P} = \mathrm{P}$; or we may say $t^n = 1$. We see also that any permutations which differ only by a power of t, belong to the same cycle.

Again, if 1 is subtracted from each of the numbers in our permutation, and the 0 in the result is replaced by n, we shall get a new permutation. For instance, from the permutation 123456, we get in this way 612345; and from 124653, we get 613542. In the former of these cases, but not in the latter, the new permutation belongs to the same cycle as the old one. By performing this process $(n-1$ times we get $(n-1)$ fresh permutations, or n in all.

Let s denote this operation, so that

$$s\mathrm{P} = (a_1 - 1, a_2 - 1, \ldots\ldots\ldots a_n - 1)$$

where, in accordance with what was said above, the constituent 0 is to be replaced by n; then the n permutations will be denoted by

$$P,\ sP,\ s^2P \ldots\ldots\ldots s^{n-1}P.$$

These will all be different from each other, but the above example shows that they may all belong to the same cycle. If we perform the operation n times, we evidently get the original permutation again, so that $s^n = 1$.

If now we combine the two kinds of operations in all possible ways, we have n^2 permutations, as follows :—

$$\left.\begin{array}{cccc} P, & tP, & t^2P, \ldots\ldots\ldots\ldots & t^{n-1}P\,; \\ sP, & stP, & st^2P, \ldots\ldots\ldots\ldots & st^{n-1}P\,; \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ s^{n-1}P, & s^{n-1}tP, & s^{n-1}t^2P, \ldots\ldots & s^{n-1}t^{n-1}P. \end{array}\right\} \text{A.}$$

The n^2 permutations thus related I call a "set". It is to be observed that the permutations are not necessarily all different. In fact, the example taken above has shown that the set may contain only n different permutations, all belonging to the same cycle.

If $P = (12435)$, we obtain in this way the following set of 25 permutations :—

12435	24351	43512	35124	51243
51324	13245	32451	24513	45132
45213	52134	21345	13452	34521
34152	41523	15234	52341	23415
23541	35412	54123	41235	12354.

Instead of writing down the 25 permutations in full, as above, we may represent them briefly as follows :—

124351243
513245132
452134521
341523415
235412354.

Here any five consecutive members constitute one of our 25 permutations; and this arrangement enables us to write down with very great ease the symbolical representation of any permutation. Thus, from $P = (12435)$, we get

$$(13245) = stP\,;\ (13452) = s^2t^3P\,;\ (15234) = s^3t^2P\,;\ (12354) = s^4t^4P$$

A little consideration shows that our s and t operations, being entirely independent of each other, may be combined like ordinary algebraical quantities, and groupt and transposed in any way we please; thus $st = ts$; $sts = s^2t = ts^2$; $(st)^2 = s^2t^2 = t^2s^2$; $s^ht^k = t^ks^h$; also $s^hs^k = s^{h+k}$; $t^ht^k = t^{h+k}$.

There is no difficulty in interpreting negative indices; thus t^{-1} denotes the operation of taking the last number in a permutation, and putting it first, or

$$t^{-1}\mathrm{P} = (a_n, a_1, a_2, \ldots\ldots\ldots a_{n-1}).$$

The permutation thus got is the same thing as $t^{n-1}\mathrm{P}$, and this follows at once from the equation $t^n = 1$; for $t^{-1}\mathrm{P} = t^nt^{-1}\mathrm{P} = t^{n-1}\mathrm{P}$.

Again, s^{-1} denotes the operation of adding 1 to each number in a permutation, substituting, however, 1 for $(n+1)$; thus

$$s^{-1}\mathrm{P} = (a_1 + 1, a_2 + 1, \ldots\ldots\ldots a_n + 1),$$

where the constituent $(n+1)$ is to be replaced by 1. We see, in fact, that n must be added (or subtracted) at each operation, as may be found necessary; and this is to be understood in future, as I shall not mention it again.

We have also $s^{-1}\mathrm{P} = s^ns^{-1}\mathrm{P} = s^{n-1}\mathrm{P}$.

We may perform the s operation on a cycle; thus,

$$s(1245312\ldots) = (5134251\ldots) = (1342513\ldots),$$

and $$s(1234512\ldots) = (5123451\ldots) = (1234512\ldots).$$

In the latter case the s operation reproduces the original cycle, and this relation may be represented by $s(\mathrm{C}) = \mathrm{C}$; and such a cycle may be called a self-repeating cycle.

In the example given above, of the set of 25 permutations derived from (12435), it is obvious that, whichever of the 25 permutations we take as our original one, we shall get by the process the same 25, but differently arranged; and the same is clearly true whatever the value of n. In other words, the set remains the same, whichever of the permutations in it is taken as the original one. In this example the 25 permutations are all different; but if we had started with the permutation (12345), or with (15432), we should have got a set containing only five different permutations, which all belong to the same cycle. Thus, putting P for (12345), we have $s\mathrm{P} = (51234)$; $t\mathrm{P} = (23451)$; $ts\mathrm{P} = st\mathrm{P} = (12345) = \mathrm{P}$. In general, since all the permutations in each column of our scheme (A) are different, and all the permutations in each row are different, a set must contain at least n different permutations; and the above example shows that it may contain n^2.

It is clear that no permutation can belong to two different sets; for if a permutation in one set is the same as a permutation in another set, it follows that all the permutations in the one set are the same as all the permutations in the other, or the two sets differ only in the arrangement of the permutations. We thus see that the $n!$ permutations of n numbers, admit of being groupt in sets containing n^2 each, so that no permutation shall occur in two of the sets. But, when n is prime, $n!$ is not divisible by n^2, and it follows that one, at least, of the sets must be such that the same permutation occurs more than once in it.

If one permutation in a set is repeated, then every permutation in it must be repeated. For instance, let the original permutation, P, be equal to $s^h t^k\mathrm{P}$, and let $s^p t^q\mathrm{P}$ be any other permutation in the set; then since $\mathrm{P} = s^h t^k\mathrm{P}$, operating on both sides with $s^p t^q$, we have

$$s^p t^q\mathrm{P} = s^p t^q(s^h t^k\mathrm{P}) = s^{p+h} t^{q+k}\mathrm{P} = s^h t^k(s^p t^q\mathrm{P}):$$

and this shows that $s^p t^q\mathrm{P}$ is repeated in the set. Each permutation in the set must, of course, be repeated the same number of times; and it follows that the number of different permutations in a set must be a divisor of n^2, and the number of different cycles in a set must be a divisor of n. If n is prime, the set must therefore contain either n permutations belonging to 1 cycle, or n^2 permutations belonging to n cycles; and such sets may be conveniently called 1-cycle sets and n-cycle sets, respectively; the permutations contained in them being similarly called 1-cycle permutations and n-cycle permutations.

It thus appears that sets, cycles, and permutations, may each be divided into two classes. We may have (1) sets in which each permutation is repeated a certain number of times; and these sets may be called "repeating sets". The permutations contained in them, and the cycles to which they belong, may be called "recurring". We may have (2) sets which contain n^2 different permutations, which belong to n different cycles; and these sets, which we have called n-cycle sets, may be also called "full sets", or "non-repeating sets". The permutations contained in these sets, which we have called n-cycle permutations, and the cycles to which they belong, may be also called "non-recurring".

I propose now to investigate the conditions that must be satisfied, in order that a permutation may belong to the one class or to the other; and I begin with ascertaining the condition that sP may belong to the same cycle as P, or that we may have $s(C) = C$. In this case, P and sP, since they belong to the same cycle, can only differ by a power of t, so that we must have $st^fP = P$, where f is an integer. We then get $s^2t^{2f}P = st^f(st^fP) = st^fP = P$; and, similarly, $P = st^fP = s^2t^{2f}P = \ldots\ldots\ldots\ldots\ldots = s^{n-1}t^{(n-1)f}P$. This shows that all the n^2 permutations in the set belong to the same cycle.

We next notice that f must be prime to n; for suppose that they are not prime, but have a common factor x, so that $f = Fx$, $n = Nx$, then, since $N < n$, we have $s^Nt^{Nf} = P$.

Also $\qquad s^Nt^{Nf}P = s^Nt^{NxF}P = s^Nt^{nF}P = s^NP$, since $t^n = 1$.

But since $s^Nt^{Nf}P = P$, it follows that $s^NP = P$; and this being impossible, f and n cannot have any common factor, or they are prime to each other.

Again, since f is prime to n, it is well known that the products, f, $2f$, $3f$,........., $(n-1)f$, when divided by n, give remainders which form a permutation of 1,2,3,.........$(n-1)$; hence there is one of the products, gf, such that $gf = qn + 1$, q being an integer, and both q and g being $< n$; and this equation shows that g, as well as f, is prime to n. Then $s^gt^{gf}P = s^gt^{qn+1}P = s^gtP$; and consequently $s^gtP = P$ or $tP = s^{-g}P$.

Hence $\qquad (a_2, a_3, \ldots\ldots\ldots a_n, a_1) = (a_1 + g, a_2 + g \ldots\ldots\ldots a_n + g)$,

and $\qquad a_2 = a_1 + g,\ a_3 = a_2 + g, \ldots\ldots\ldots, a_1 = a_n + g$;

or each constituent in P is got by adding g to the preceding one, or subtracting $(n - g)$ from it; and

$$P = \{a_1, a_1 + g, a_1 + 2g, \ldots\ldots\ldots a_1 + (n-1)g\}.$$

Conversely, if we take any number a not $> n$, and g any number $< n$ and prime to it, since the numbers a, $a + g$, $a + 2g$,......... $a + (n-1)g$, all give different remainders when divided by n, the above formula for P will give a permutation of the numbers 123 ....n. It is obvious from the formula, that $s^gtP = P$; and, by reversing the above demonstration, we may show that

$$P = st^fP = s^2t^{2f}P = \ldots\ldots\ldots = s^{n-1}t^{(n-1)f}P\,;$$

whence P is a 1-cycle permutation.

When n is prime, we may give g any one of the values 1,2,3, ...$(n-1)$, and there are therefore $(n-1)$ 1-cycle sets, and $(n-1)$ self-repeating cycles, which may be obtained from the $(n-1)$ permutations.

$$\begin{array}{lllll}
\{1, & 1+1, & 1+2, & \ldots\ldots\ldots & 1+\ (n-1)\} \\
\{1, & 1+2, & 1+4, & \ldots\ldots\ldots & 1+2(n-1)\} \\
\{1, & 1+3, & 1+6, & \ldots\ldots\ldots & 1+3(n-1)\} \\
\multicolumn{5}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\
\{1, & 1+(n-1), & 1+2(n-1), & \ldots\ldots\ldots & 1+\ (n-1)^2\}.
\end{array}$$

When n is a composite number, the number of such sets is $\phi(n)$, or the number of numbers $<n$ and prime to it.

Next take the more general case, and suppose that $s^h t^k P = P$; then we can prove that, either h and k are both prime to n; or, if x is the G. C. M. of h, k, n, and $h = xH$, $k = xK$, $n = xN$, then H and K are both prime to N. If h and k are not both prime to n, suppose that h and n contain a common factor x, so that $h = xH$, $n = xN$; then $s^h t^k = s^{xH} t^k$, and $(s^h t^k)^N = s^{NxH} t^{Nk} = s^{nH} t^{Nk} = t^{Nk}$. Hence $s^h t^k P = P$ leads to $t^{Nk} P = P$; but this cannot be the case unless Nk is a multiple of n or of Nx, and therefore k must be a multiple of x. Hence every common factor which h and n contain, must be also a factor of k; and it may be proved similarly that every common factor which k and n contain, must be also a factor of h. Hence, if we divide by the G. C. M. of h, k, n, suppose x, the quotients, H and K, must both be prime to N; for, if this were not the case, but H and N, for instance, contained a common factor y, then h and n would have the common factor xy, which is not a factor of k. Since h and k are both $<$ than n, H and K are both $<$N. It follows that, if $h = 1$, or $st^k P = P$, then k must be prime to n; and if $k = 1$, or $s^h t P = P$, then h must be prime to n.

We will first consider the case where h and k are both prime to n. We have

$$P = s^h t^k P = s^{2h} t^{2k} P = \ldots\ldots\ldots\ldots = s^{(n-1)h} t^{(n-1)k} P,$$

and we have therefore to consider the operations

$$s^h t^k,\ s^{2h} t^{2k},\ s^{3h} t^{3k} \ldots\ldots\ \ldots\ \ldots s^{(n-1)h} t^{(n-1\ k},$$

and to suppress the multiples of n that occur in the indices. Now, h being prime to n, if we divide n into the numbers, h, $2h$, $3h$......... $(n-1)h$, we shall have $(n-1)$ different remainders, one of which must therefore be 1; and the same is true of k; therefore the above operations contain two which are equivalent to st^f, $s^g t$, so that we have $P = st^f P = s^g t P$. Hence, by what was proved above, both f and g must be prime to n, and P must be a 1-cycle permutation, or must be of the form

$$\{a,\ a+g,\ a+2g, \ldots\ldots\ldots\ldots\ldots a+(n-1)g\}.$$

It is to be observed that, if $s^h t^k P = P$, and P is of the above form, it does not follow that h and k are prime to n; and it is easy to see that this will not always be the case. For instance, if $n = 6$, and $stP = P$, so that P is of the form $(a, a+1, a+2, \ldots\ldots\ldots\ldots a+5)$, we shall have $s^2t^2P = P$, $s^3t^3P = P$, $s^4t^4P = P$.

When n is prime, h and k are necessarily prime to it; and, whatever values we give to h and k, we shall get by the above process a value of g. Since we may give h any of the $(n-1)$ values $1,2,3 \ldots\ldots\ldots (n-1)$, and the same is true of k, the total number of pairs of values of h and k is $(n-1)^2$; and as there can only be $(n-1)$ values of g, or $(n-1)$ self-repeating cycles, it follows that $(n-1)$ of the pairs belong to each cycle.

We are now in a position to determine, in the case when n is prime, how many permutations and sets there are of each class. We have seen that there are $(n-1)$ 1-cycle sets, and these contain $n(n-1)$ recurring permutations. Also, the total number of permutations being $n!$, the number of non-recurring permutations is $n! - n(n-1) = n(n-1)\{(n-2)! - 1\}$.

Each of these, as we have seen, belongs to a set which contains n^2 different permutations; hence the number of these sets is

$$n(n-1)\{(n-2)! - 1\} \div n^2 = \frac{n-1}{n}\{(n-2)! - 1\}.$$

This number is always integral; for by Wilson's Theorem, $(n-1)! + 1$ is divisible by n. Suppose the quotient to be Q, so that $(n-1)! + 1 = Qn$; then $(n-1) \times (n-2)! + 1 = Qn$, and $(n-2)! - 1 = n\{(n-2)! - Q\}$.

We thus see that, when n is prime, there are $n(n-1)\{(n-2)! - 1\}$ n-cycle permutations, which can be arranged in n-cycle sets; and $n(n-1)$ 1-cycle permutations, which belong to $(n-1)$ 1-cycle sets. And since each cycle gives n permutations, we see that the total number of cycles is $(n-1)!$; also that $(n-1)$ of these are self-repeating, and the remaining $(n-1)\{(n-2)! - 1\}$ are non-recurring, or are such that the s operation gives a new cycle in every case.

When $n = 5$, and we give g the values 1, 2, 3, 4, we get the four following self-repeating cycles :—

1234512...... ; 1352413...... ; 1425314...... ; 1543215...... ;

It will be found by actual trial that the 20 non-recurring cycles are :—

C	1235412......	1245312......	1254312......	1354213......
sC	1243512......	1342513......	1432514......	1524315......
s^2C	1324513......	1452314......	1453214......	1325413......
s^3C	1345213......	1253412......	1534215	1435214......
s^4C	1523415......	1423514......	1542315......	1532415......

These results may be otherwise stated by saying that, when we form all the possible permutations by means of the s and t operations, the following are 1-cycle permutations, and give rise to 5 permutations each, or 20 in all :—

12345, 13524, 14253, 15432;

and the following are 5-cycle permutations, and give rise to 25 each, or 100 in all :—

12354, 12453, 12543, 13542.

It will be convenient now to show how to find f and g when we know h, k, and n; h and k being both prime to n. We have seen that there exists some number g less than n, such that $s^g t$ is equivalent to $s^{mh}t^{mk}$, where $m<n$. It follows that $mk=qn+1$, where q is some integer which must be less than n, because both m and k are less than n; also $mh=q'n+g$. If for any value of n we form a table of the following form, we shall be able by inspection to determine the value of m when we know k.

Table showing the values of $(qn+1)$, and its resolution into two factors, each less than n.

q	$n=11$	$n=12$
	$11q+1$	$12q+1$
0	$1=1\times 1$	$1=1\times 1$
1	$12=2\times 6=3\times 4$	13
2	23	$25=5\times 5$
3	34	37
4	$45=5\times 9$	$49=7\times 7$
5	$56=7\times 8$	61
6	67	73
7	78	85
8	89	97
9	$100=10\times 10$	109
10	111	$121=11\times 11$
11		133

We see from this table that, when $n=11$ and 12 respectively, and $q<n$, the numbers $(qn+1)$ can in certain cases be resolved into pairs of factors, each $<n$; and that these factors include all the numbers $<n$ and prime to it. It is easy to prove that this will always be the case. If k is any number $<n$ and prime to it, and we divide n into the $(n-1)$ numbers k, $2k$, $3k$,.........$(n-1)k$, we shall get $(n-1)$ different remainders, one of which must therefore be 1 : hence there is some number $m<n$, such that mk is of the form $(qn+1)$. This proves the proposition.

If n is a largeish number, the table is easily formed with the help of a table of the prime factors of numbers. In this way I have got the following results for $n=101$ and 120.

q	$n=101$	
	$qn+1$	Factors.
0	1	1×1
1	102	2×51, 3×34, 6×17
2	203	7×29
3	304	4×76, 8×38, 16×19
4	405	5×81, 9×45, 15×27
5	506	11×46, 22×23
7	708	12×59
9	910	10×91, 13×70, 14×65, 26×35
13	1314	18×73
16	1617	21×77, 33×49
19	1920	20×96, 24×80, 30×64, 32×60, 40×48
20	2021	43×47
22	2223	39×57
23	2324	28×83
24	2425	25×97
26	2627	37×71
27	2728	31×88, 44×62
28	2829	41×69
31	3132	36×87, 54×58
32	3233	53×61
35	3536	52×68
37	3738	42×89
49	4950	50×99, 55×90, 66×75
51	5152	56×92
58	5859	63×93
61	6162	78×79
63	6364	74×86
65	6566	67×98
67	6768	72×94
69	6970	82×85
79	7980	84×95
99	10000	100×100

q	$n=120$	
	$qn+1$	Factors.
0	1	1×1
1	121	11×11
3	361	19×19
4	481	13×37
6	721	7×103
7	841	29×29
8	961	31×31
9	1081	23×47
14	1681	41×41
16	1921	17×113
20	2401	49×49
24	2881	43×67
29	3481	59×59
31	3721	61×61
34	4081	53×77
42	5041	71×71
52	6241	79×79
59	7081	73×97
66	7921	89×89
69	8281	91×91
74	8881	83×107
85	10201	101×101
99	11881	109×109
118	14161	119×119

The same table will enable us to find f when we know g, or to find g when we know f; for the relation between f and g is exactly the same as that between k and m.

The table shows (1) the values of $(qn+1)$ for all values of $q<n$;

and (2) the factors of $(qn+1)$ when it is the product of two numbers each $<n$. Thus, when $n=11$, the table shows us that

if k (or f) $=$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10;
then m (or g) $=$ 1, 6, 4, 3, 9, 2, 8, 7, 5, 10.

Again, when $n=12$,

if k (or f) $=$ 1, 5, 7, 11;
then m (or g) $=$ 1, 5, 7, 11.

Having got m, the equation $mh=q'n+g$, shows that g is the remainder we get when mh is divided by n. Suppose, for example, that $n=11$, and $h=2$, $k=3$; so that $s^2t^3\mathrm{P}=\mathrm{P}$. Then, from above, $m=4$, $mh=8$; so that $g=8$; and then $f=7$, or $st^7\mathrm{P}=s^8t\mathrm{P}=\mathrm{P}$. Hence P must be a permutation belonging to the cycle 1,9,6,3,11,8,5,2,10,7,4,1,9.........where each constituent is got from the preceding by adding 8 or subtracting 3; and a little consideration shows that P may be any such permutation.

The positions in the scheme (A) of the 10 permutations which are identical with the original one, are given by the following values of the indices of s and t, which correspond to st^7, s^2t^{14}, s^3t^{21}, etc.

1,7; **2,3**; 3,10; 4,6; 5,2; 6,9; 7,5; 8,1; 9,8; 10,4;

or they may be otherwise arranged, so as to correspond with s^8t, $s^{16}t^2$, etc.

8,1; 5,2; **2,3**; 10,4; 7,5; 4,6; 1,7; 9,8; 6,9; 3,10.

But their relation to each other will be better understood by observing the positions of the identical numbers in a series of permutations where each is got from the one above it by the s operation. Retaining the value $g=8$, let us take 8 as the initial constituent of the permutation; then the 11 permutations in question are:—

8	5	2	10	7	4	1	9	6	3	11
7	4	1	9	6	3	11	**8**	5	2	10
6	3	11	**8**	5	2	10	7	4	1	9
5	2	10	7	4	1	9	6	3	11	**8**
4	1	9	6	3	11	**8**	5	2	10	7
3	11	**8**	5	2	10	7	4	1	9	6
2	10	7	4	1	9	6	3	11	**8**	5
1	9	6	3	11	**8**	5	2	10	7	4
11	**8**	5	2	10	7	4	1	9	6	3
10	7	4	1	9	6	3	11	**8**	5	2
9	6	3	11	**8**	5	2	10	7	4	1

If, instead of taking $s^2t^3P = P$, we had taken values of h and k corresponding to any other of the identical permutations, we should have got the same result; for instance, if $s^5t^2P = P$; or $h = 5$, $k = 2$; we have $m = 6$, $mh = 30 = 2 \times 11 + 8$; and $g = 8$, as before.

As mentioned above, there are $(n-1)^2$, or 100, pairs of values of h and k, to each of which corresponds a value of g; and in the adjoining diagram each of the squares with a number in it, represents one of these pairs of values, and the number in it is the value of g for that pair.

$k = $**10**		10	9	8	7	6	5	4	3	2	1
9		5	10	4	9	3	8	2	7	1	6
8		7	3	10	6	2	9	5	1	8	4
7		8	5	2	10	7	4	1	9	6	3
6		2	4	6	8	10	1	3	5	7	9
5		9	7	5	3	1	10	8	6	4	2
4		3	6	9	1	4	7	10	2	5	8
3		4	8	1	5	9	2	6	10	3	7
2		6	1	7	2	8	3	9	4	10	5
1		1	2	3	4	5	6	7	8	9	10
0											
	$h = $**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

Next take $n = 12$, and $h = 5$, $k = 11$; then from above $m = 11$, $mh = 55 = 4 \times 12 + 7$, or $g = 7$; whence, again, $f = 7$. Therefore P belongs to the cycle

1,8,3,10,5,12,7,2,9,4,11,6,1,8.........

Also the positions of the 11 permutations that are identical with the original one, are indicated by

1,7 ; 2,2 ; 3,9 ; 4,4 ; **5,11** ; 6,6 ; 7,1 ; 8,8 ; 9,3 ; 10,10 ; 11,5.

Here the following pairs of values of h and k are prime to $n(=12)$; namely, 1,7 ; 5,11 ; 7,1 ; 11,5 ; and taking any one of these pairs, we should by the same process find the same values of f and g. In the remaining pairs, 2,2 ; 3,9 ; 4,4 ; 6,6 ; 8,8 ; 9,3 ; 10,10 ; we see that, consistently with what was said above, both numbers in each pair contain a factor which is also a factor of 12 ; and dividing by the G.C.M.'s, 2, 3, 4, 6, 4, 3, 2 respectively, the quotients are both prime to the quotient of 12 ; for instance, taking 3,9, and dividing by the G.C.M., 3, we get 1,3, each of which is prime to the quotient of 12 by 3, namely 4.

This example shows us that, although when h,k, are both prime to n, the relation $s^h t^k \mathrm{P} = \mathrm{P}$, cannot be satisfied unless P is a 1-cycle permutation, yet when P is such a permutation, and $s^h t^k \mathrm{P} = \mathrm{P}$, it is not necessary that h, k, should be prime to n, but only that, when all three numbers are divided by their G.C.M., the quotients of h,k, should be prime to the quotient of n.

When $n=12$, and h,k, are both prime to it, the only possible pairs of values of h and k are the following 16 :—

1,1 ; 1,5 ; 1,7 ; 1,11 ;   5,1 ; 5,5 ; 5,7 ; 5,11 ;
7,1 ; 7,5 ; 7,7 ; 7,11 ;   11,1 ; 11,5 ; 11,7 ; 11,11.

Each of these belongs to a 1-cycle permutation ; and, taking any pair, we can, by the process above described, find the value of g. Since g must be prime to 12, it can only have the 4 values 1, 5, 7, 11 ; so that there can be only 4 1-cycle permutations, and 4 of the pairs of values of h,k, belong to each of these. The pairs that belong to each value of g are shown in the adjoining diagram.

k=11		11				7		5				1
10			11, 5								1, 7	
9				11, 7						1, 5		
8					11, 5				1, 7			
7		7				11		1				5
6							11, 7 1, 5					
5		5				1		11				7
4					1, 7				11, 5			
3				1, 5						11, 7		
2			1, 7								11, 5	
1		1				5		7				11
0												
h=	0	1	2	3	4	5	6	7	8	9	10	11

For the sake of completeness I have inserted the values of g that belong to pairs of values of h and k that are not prime to 12; and it will be noticed that every one of such pairs belongs to more than one cycle. We shall see later on how the values of g may be determined in these cases. The squares which are left blank in the diagram, are those which correspond to values of h and k which cannot coexist.

Having thus fully considered the case where h and k are prime to n, we have now to consider the case where all 3 numbers have a common factor. Suppose that x is the G.C.M., and $h = x\text{H}$, $k = x\text{K}$, $n = x\text{N}$; then, as we have seen, H, K, are both prime to N, and, of course, $<$N. Hence among the numbers H, 2H, 3H......(N - 1)H, there is one which gives the remainder 1 when divided by N; and the same is true of the numbers K, 2K, 3K,......(N - 1)K.
Hence the operations

$$(s^{H}t^{K})^{x},\ (s^{H}t^{K})^{2x},\ (s^{H}t^{K})^{3x}\ldots\ldots(s^{H}t^{K})^{(N-1)x}$$

contain two which are equivalent to $(s^{mN+1}t^{F})^{x}$ and $(s^{G}t^{mN+1})^{x}$, or to $(st^{F})^{x}$ and $(s^{G}t)^{x}$, since $Nx=n$, and $t^{n}=1$. Here F and G must both be $<$N, and prime to it; and they are to be found from H,K, in the same way that f,g, were found from h,k.

We next observe that $P=(st^{F})^{x}P$ leads to $P=s^{x}t^{Fx}P=s^{2x}t^{2Fx}P=\ldots\ldots=s^{(N-1)x}t^{(N-1)Fx}P$. This shows that these N permutations all belong to the same cycle, and it follows that there cannot be more than x different cycles in the set to which they belong. We shall presently see that there may be fewer.

Resuming now the relation $(s^{G}t)^{x}P=P$, or $t^{x}P=s^{-Gx}P$, we have

$$t^{x}P=(a_{x+1},a_{x+2},\ldots\ldots a_{n},a_{1},a_{2},\ldots\ldots a_{x})$$

and
$$s^{-Gx}P=(a_{1}+Gx,a_{2}+Gx,a_{3}+Gx\ldots\ldots a_{n}+Gx)$$

and it follows from the identity of these that

$$\begin{aligned}
&a_{x+1}=a_{1}+Gx,\ a_{2x+1}=a_{x+1}+Gx,\ldots\ldots\\
&a_{x+2}=a_{2}+Gx,\ a_{2x+2}=a_{x+2}+Gx,\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&a_{2x}=a_{x}+Gx,\ a_{3x}=a_{2x}+Gx,\ldots\ldots\ldots
\end{aligned}$$

whence $a_{2x+1}=a_{1}+2Gx,\ a_{2x+2}=a_{2}+2Gx,\ldots\ldots\ldots\ldots$
and generally $a_{mx+1}=a_{1}+mGx,\ a_{mx+2}=a_{2}+mGx,\ldots\ldots\ldots\ldots$
Hence we get

$$P=\{a_{1},a_{2},a_{3}\ \ldots\ldots a_{x},a_{1}+Gx,a_{2}+Gx\ldots\ldots a_{x}+Gx,a_{1}+2Gx\ldots\ldots$$
$$a_{x}+2Gx,\ldots\ldots\ldots\ldots a_{1}+(N-1)Gx,\ldots\ldots\ldots\ldots a_{x}+(N-1)Gx\}$$

Here we know G, which has been found from H and K; and our next step will be to show how many permutations of this form can be got from any known value of G. We first observe that, if a_{1}, a_{2}, $\ldots\ldots a_{x}$, are x consecutive numbers, all less than n, then all the numbers in the above expression for P will give different remainders when divided by n. For any two of these numbers may be represented by $a_{u}+mGx$, $a_{v}+m'Gx$, where a_{u} and a_{v} are two of the numbers, $a_{1},a_{2},\ldots\ldots a_{x}$, and therefore $v-u<x$. Suppose these two numbers, if possible, to give the same remainder, R, so that

$$a_{u}+m\,Gx=qn+R=q\,Nx+R,$$
$$a_{v}+m'Gx=q'n+R=q'Nx+R;$$

then
$$a_{u}-a_{v}+(m-m')Gx=(q-q')Nx.$$

In order that this equation may subsist, $a_u - a_v$ must be divisible by x; but this is impossible, because the difference between a_u and a_v is $<x$, these being two of x consecutive numbers. Hence no two of the n numbers can give the same remainder, or they all give different remainders. If, then, we take any x consecutive numbers, $<n$, and form P according to the above formula, we shall get a permutation of $1,2,3,\ldots\ldots n$. It is not necessary, however, that $a_1,a_2 \ldots\ldots a_x$ should be consecutive numbers; but if, in place of any one of them, a_u, we set any one of the numbers $a_u + Gx$, $a_u + 2Gx, \ldots\ldots a_u + (N-1)Gx$, and form P by the same formula, we shall get another permutation satisfying the same condition. In this way from any one permutation we get $(N-1)$ others, or N in all. But each of the x numbers $a_1,a_2 \ldots\ldots a_x$, may be separately dealt with in the same way; and thus we get in all N^x different permutations satisfying the condition. Lastly, the x numbers $a_1,a_2 \ldots\ldots a_x$, may be permuted in $x!$ ways; and for each of these arrangements we get, as we have seen, N^x different permutations, so that, finally, we have $x! \times N^x$ different permutations of $1,2,3,\ldots\ldots n$, which satisfy the required condition.

This result may also be arrived at as follows:—

Having assumed a_1 at pleasure, we must have $a_{x+1} = a_1 + Gx$, $a_{2x+1} = a_1 + 2Gx, \ldots\ldots a_{(N-1)x+1} = a_1 + (N-1)Gx$; so that N of the numbers in the required permutation are determined. This leaves $n - N = N(x-1)$ numbers, any one of which may he taken for a_2, and thus N more of the numbers in the permutation are determined. This leaves $n - 2N = N(x-2)$ numbers, any one of which may be taken for a_3, and so on. Hence, bearing in mind that we may take any one of the n (or Nx) numbers for a_1, the total number of permutations for a known value of G is

$$Nx.N(x-1).N(x-2)\ldots\ldots N(1) = N^x \times x!$$

As an example, let us take $n = 12$, $h = 3$, $k = 9$, so that $x = 3$, $N = 4$, $H = 1$, $K = 3$: then we have the auxiliary table annexed. This shows that $m = 3$; then $mH = 3$, and $G = 3$, and P must be of the form, (suppressing multiples of 12)

$$(a_1, a_2, a_3, a_1 + 9, a_2 + 9, a_3 + 9, a_1 + 6, a_2 + 6, a_3 + 6, a_1 + 3, a_2 + 3, a_3 + 3).$$

$N = 4$	
q	$4q+1$
0	$1 = 1^2$
1	5
2	$9 = 3^2$
3	13

Giving a_1,a_2,a_3, the values 1,2,3, we may conveniently represent P in the following way :

$$\left|\begin{array}{c} 1.10.7.4 \\ 2.11.8.5 \\ 3.12.9.6 \end{array}\right|$$

where it is to be understood that the figures in each line may be cyclically transposed, independently of those in the other lines, and the order of the lines may also be altered as we please. All the permutations thus got will satisfy the condition $s^3t^9\mathrm{P} = \mathrm{P}$, or $s^3\mathrm{P} = t^3\mathrm{P}$; and the total number of such permutations is $4^3 \times 3! = 384$, belonging to 32 cycles.

If we take, for instance 1,9,11, as the first 3 numbers, we have the permutation

1, 9, 11, 10, 6, 8, 7, 3, 5, 4, 12, 2;

and then performing the s operation 3 times, we get

12, 8, 10, 9, 5, 7, 6, 2, 4, 3, 11, 1,
11, 7, 9, 8, 4, 6, 5, 1, 3, 2, 10, 12,
10, 6, 8, 7, 3, 5, 4, 12, 2, 1, 9, 11;

whence we see that the permutation satisfies the condition $s^3t^9\mathrm{P} = \mathrm{P}$, or $s^3\mathrm{P} = t^3\mathrm{P}$; also that it is a 3-cycle permutation.

All the 384 will not be 3-cycle permutations ; for the following are 1-cycle permutations, corresponding to the values of g, 7 and 11,

1, 8, 3, 10, 5, 12, 7, 2, 9, 4, 11, 6;
1, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2.

The 1-cycle permutations which satisfy the condition $(s^{\mathrm{G}}t)^x\mathrm{P} = \mathrm{P}$, may be determined as follows :—

This condition will be satisfied by $s^{\mathrm{G}}t\mathrm{P} = \mathrm{P}$ if this is a possible condition, that is to say, if G is prime to n, or prime to x as well as N. In the above example, G = 3, and is not prime to n, which is 12. But, since $s^n = s^{x\mathrm{N}} = 1$, the condition $\mathrm{P} = (s^{\mathrm{G}}t)^x\,\mathrm{P}$ leads to

$$\mathrm{P} = (s^{\mathrm{G}}t)^x\mathrm{P} = (s^{\mathrm{G+N}}t)^x\mathrm{P} = (s^{\mathrm{G+2N}}t)^x\mathrm{P} = \dots\dots\dots = \{s^{\mathrm{G}+(x-1)\mathrm{N}}t\}^x\mathrm{P}\,;$$

and from this we see that the condition will be satisfied if the index of s, which may as usual be denoted by g, has any one of the values G,G + N,G + 2N,......G + $(x - 1)$N, provided it is prime to n, or prime to x as well as N. If x is prime to N, one of these x numbers is divisible by x, and the others give different remainders when divided by x ; and if x is a prime, these $(x - 1)$ remainders will all be prime to it, so that in the latter case we have $(x-1)$ different self-repeating cycles to which P may belong.

In the example we have been considering, $x=3$, $N=4$, $G=3$; and g may have the values $G+N, G+2N$, or 7 and 11; and we get the 2 self-repeating cycles given above.

In the case of these self-repeating cycles, the first x constituents of the cycle become

$$\begin{array}{ll} a\,, a_1+G, & a_1+2G, \ldots\ldots\ldots a_1+(x-1)G \\ \text{or } a_1, a_1+G+N, & a_1+2(G+N) \ldots\ldots a_1+(x-1)(G+N) \\ \text{or } a_1, a_1+G+2N, & a_1+2(G+2N) \ldots\ldots a_1+(x-1)(G+2N) \\ \text{or} \quad \text{etc.} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

or

$$a_1, a_1+G+(x-1)N, a_1+2\{G+(x-1)N\}, \ldots a_1+(x-1)\{G+(x-1)N\}$$

We have thus seen that the permutations which satisfy the condition $(s^G t)^x P = P$ may be either x-cycle permutations or 1-cycle permutations. But if x is a composite number, $=yz$ suppose, there may be also y-cycle and z-cycle permutations. In order that they may be y-cycle permutations, we must give $a_{1+y}, a_{1+2y} \ldots\ldots a_{1+(z-1)y}$ the same values as they have in the 1-cycle permutations, as shown in the following scheme :—

	either	or	etc., etc.
a_{1+y}	a_1+yG	$a_1+y(G+N)$	
a_{1+2y}	a_1+2yG	$a_1+2y(G+N)$	
…… ……	………………	……………………	
$a_{1+(z-1)y}$	$a_1+(z-1)yG$	$a_1+(z-1)y(G+N)$	

For instance, if $n=12$, $x=4$, $N=3$, suppose we have found $G=4$; then the permutations will be represented by

1.5. 9
2.6.10
3.7.11
4.8.12

Here g may have the values 1,7; and we have 1-cycle permutations, belonging to the cycles :—

1,2,3, 4,5, 6,8,9,10,11,12,1.........
1,8,3,10,5,12,7,2, 4,11, 6,1........

We have also 2-cycle permutations, which may be represented by

$$\left|\begin{array}{l}1.3.5.7.\ 9.11\\ 2.4.6.8.10.12\end{array}\right|$$

Thus the permutation 1,4,3,6,5,8,7,10,9,12,11,2, is a 2-cycle permutation.

If, instead of supposing h and k to be known, and therefore H and K, we simply suppose n to be a composite number $= x\mathrm{N}$, it may be proved by a similar process that, if G be any number $< \mathrm{N}$ and prime to it, the permutations given by the formula on p. 72 all satisfy the condition $(s^{\mathrm{G}}t)^x\mathrm{P} = \mathrm{P}$.

Resuming our example, $n = 12$, $x = 3$, $\mathrm{N} = 4$, we may give G the values 1 and 3, and the permutations that satisfy the conditions $(st)^3\mathrm{P} = \mathrm{P}$, and $(s^3t)^3\mathrm{P} = \mathrm{P}$ respectively, are represented by

$$\left|\begin{array}{l}1.4.7.10\\ 2.5.8.11\\ 3.6.9.12\end{array}\right| \text{ and } \left|\begin{array}{l}1.10.7.4\\ 2.11.8.5\\ 3.12.9.6\end{array}\right| .$$

When $\mathrm{G} = 1$, we may give g the values 5 and 1; and when $\mathrm{G} = 3$, we may give g the values 7 and 11.

Next putting $n = 12$, $x = 4$, $\mathrm{N} = 3$, we may give G the values 1 and 2, and the permutations that satisfy the conditions $(st)^4\mathrm{P} = \mathrm{P}$, and $(s^2t)^4\mathrm{P} = \mathrm{P}$ respectively are represented by

$$\left|\begin{array}{l}1.5.\ 9\\ 2.6.10\\ 3.7.11\\ 4.8.12\end{array}\right| \text{ and } \left|\begin{array}{l}1.\ 9.5\\ 2.10.6\\ 3.11.7\\ 4.12.8\end{array}\right|$$

When $\mathrm{G} = 1$, we may give g the values 1 and 7; and when $\mathrm{G} = 2$, we may give g the values 5 and 11.

Again, putting $n = 12$, $x = 2$, $\mathrm{N} = 6$, we may give G the values 1 and 5; and the permutations that satisfy the conditions $(st)^2\mathrm{P} = \mathrm{P}$, and $(s^5t)^2\mathrm{P} = \mathrm{P}$ respectively, are represented by

$$\left|\begin{array}{l}1.3.5.7.\ 9.11\\ 2.4.6.8.10.12\end{array}\right| \text{ and } \left|\begin{array}{l}1.11.\ 9.7.5.3\\ 2.12.10.8.6.4\end{array}\right|$$

When $\mathrm{G} = 1$, g may be 1 or 7; and when $\mathrm{G} = 5$, g may be 5 or 11.

Lastly, putting $n = 12$, $x = 6$, $\mathrm{N} = 2$, we can only give G the value 1; and g may be 1, 5, 7, or 11.

These results agree with those that we obtained before, see p. 70.

We have seen that when $s^ht^k\mathrm{P} = \mathrm{P}$, we always have $s^{mh}t^{mk}\mathrm{P} = \mathrm{P}$, and we are now able to answer the question whether the latter condition necessarily implies the former. This will always be the case when m is prime to n; for then one of the numbers, m, $2m$, $3m$,...... $(n-1)m$ must give a remainder 1 when divided by n. For instance, when $n=11$, if we have $s^6t^8P=P$, we must also have $s^3t^4P=P$; and when $n=12$, if $s^5t^5P=P$, we must have $stP=P$. In both these cases, P must be a 1-cycle permutation. It is not necessary that h and k should be prime to n; thus, if $n=12$, $h=k=2$, $m=5$, so that $s^{10}t^{10}P=P$, we must have $s^2t^2P=P$, and we may have also $stP=P$. In the latter case, P is a 1-cycle permutation for which $g=1$; but when $s^2t^2P=P$, P may either be one of the 1-cycle permutations for which $g=1$ or 7; or one of many 2-cycle permutations.

When m is not prime to n, it does not necessarily follow that $s^ht^kP=P$; but this may be the case if h and k are both prime to n, or if, on dividing all 3 numbers by their G.C.M., the quotients of h and k are both prime to the quotient of n. If h and n have a common factor which is not a factor of k, or if k and n have such a factor which is not a factor of h, the relation $s^ht^kP=P$ cannot subsist.

Let us now suppose that both x and N are primes; then, as we have seen, there are $(N-1)$ admissible values of G; and for each of these, there are $(x-1)$ admissible values of g. There are, therefore, $(N-1)(x-1)$ self-repeating cycles, and $n(x-1)(N-1)$ 1-cycle permutations which satisfy the condition $(s^Gt)^xP=P$. Now we have seen that the total number of permutations which satisfy that condition for a given value of G, is $N^x \times x!$; and the total number, for all values of G, is therefore $(N-1)N^x \times x!$ Now, since both x and N are prime, all of these must be either 1-cycle permutations or x-cycle permutations. Subtracting, then, the number of 1-cycle permutations, as found above, we get the number of x-cycle permutations:

$$\begin{aligned}&(N-1)N^x \times x! - n(N-1)(x-1)\\ &= n(x-1)(N-1)\{N^{x-1} \times (x-2)! - 1\}\end{aligned}$$

Similarly, the number of N-cycle permutations is

$$n(x-1)(N-1)\{x^{N-1} \times (N-2)! - 1\}$$

But the total number of permutations of the n numbers is $n!$, and all of these must be either 1-cycle, x-cycle, N-cycle, or n-cycle permutations. Hence, by subtraction, the number of n-cycle permutations is

$$n! - n(x-1)(N-1)\{N^{x-1} \times (x-2)! + x^{N-1} \times (N-2)! - 1\}$$

Since these can be arranged in sets containing n^2 each, this number must be divisible by n^2. Now, since $n=xN$, $(n-1)!$ is clearly divisible by n, and therefore $n!$ by n^2; and we thus see that the number will be divisible by n^2 if

$$N^{x-1} \times (x-2)! + x^{N-1} \times (N-2)! - 1 \ldots\ldots\ldots Z$$

is divisible by n. We will first prove that it is divisible by x. Since x is prime, and N prime to it, N^{x-1} is of the form $Mx+1$ (Fermat's Theorem): also $(x-2)!$ is of the form $M'x+1$ (Wilson's Theorem): hence $N^{x-1} \times (x-2)!$ is of the form $mx+1$, and $N^{x-1} \times (x-2)! - 1$ is divisible by x. And, since the other term in Z, namely, $x^{N-1} \times (N-2)!$ is divisible by x, Z itself is so divisible. Similarly Z is divisible by N, and therefore by xN or n.

These results may be otherwise stated as follows:—
Suppose that $n = xy$, where x and y are both prime, then the number of 1-cycle sets is $(x-1)(y-1)$; and this is also the number of the self-repeating cycles, while the number of the 1-cycle permutations is n times as many.

The number of the x-cycle sets is found by dividing the number of the x-cycle permutations, as above found, by nx, and is therefore

$$(x-1)(y-1)\{y^{x-1}(x-2)! - 1\} \div x$$

The number of y-cycle sets is

$$(x-1)(y-1)\{x^{y-1}(y-2)! - 1\} \div y\,;$$

and the number of n-cycle sets is

$$[(n-1)! - (x-1)(y-1)\{y^{x-1}(x-2)! + x^{y-1}(y-2)! - 1\}] \div n.$$

As an example we will take the case of $n = 6$. Here the only possible values of g are 1 and 5; and these give us the two self-repeating cycles:— 12345612........., 16543216.........
Next, putting $x = 2$, $y = 3$, we may have $G = 1$ or 2, and each of these values gives us $3^2 \times 2! = 18$ permutations. Those given by $G = 1$ may be symbolized by $\left|\begin{matrix}1.3.5\\2.4.6\end{matrix}\right|$ These belong to 3 cycles, one of which, namely, 12345612......... is self-repeating; and excluding this, and writing down the other two, and performing the *s* operation on them, we have

143652........... 163254...........
632541........... 652143...........
521436........... 541632...........
416325........... 436521...........
365214........... 325416...........
254163........... 214365...........

from which we see that the two cycles belong to a 2-cycle set.

Next, taking G = 2, we have $\begin{vmatrix} 1.5.3 \\ 2.6.4 \end{vmatrix}$, which gives us the self-repeating cycle 16543216........., and two other cycles,

12563412.........   16543216.........

which belong to another 2-cycle set.

Again, if we take $x = 3$, $y = 2$, the only admissible value of G is 1, and we get $2^3 \times 3!$ or 48 permutations symbolized by $\begin{vmatrix} 1.4 \\ 2.5 \\ 3.6 \end{vmatrix}$. These belong to 8 cycles, 2 of which are the self-repeating cycles we have already had, and the other 6 are

126453.........,   132465.........,   135462.........,
153426.........,   156423.........,   162435......... ;

and these belong to two 3-cycle sets, as follows :—

126453.........   132465.........
615342.........   621354.........
564231.........   516243.........
453126.........   465132.........
342615.........   354621.........
231564.........   243516.........

We have thus got

2 cycles belonging to 1-cycle sets
4 ,, ,, ,, 2-cycle ,,
6 ,, ,, ,, 3-cycle ,,

or 12 cycles in all ; and the total number of cycles being 5! or 120, the number of cycles which belong to 6-cycle sets is 108, and the number of such sets is 18. By actual trial I have found that the following 18 permutations give rise to these sets :—

123465
123564
123645
123654
124365
124635
124653
125364
125463
125643
126354
126435
126543
132654
135264
136254
136542
142653

On the numerical values of the roots of the equation $\cos x = x$.

By T. H. MILLER, M.A.

The angles which satisfy the equation

$$x = \cos x,$$

occur in the solution of certain problems in analytical geometry.

It is easy to show that one value of the angle is coscoscos...cos 1, when the cosine is taken successively to infinity.

The value taken ten times is ·7442, which is correct to two decimal places, but the numerical values are more easily found by the method of trial, and the following considerations facilitate the work.

First: there is only one real root. For while x increases from 0 to $\pi/2$, $\cos x$ diminishes from 1 to 0. Also $\cos x$ is positive while x lies between 0 and -1.

This root lies near $\frac{\pi}{4}$. Then let

$$\cos\left(\frac{\pi}{4} - \beta\right) = \frac{\pi}{4} - \beta$$

where β is small; that is

$$\frac{\sqrt{2}}{2}(\sin\beta + \cos\beta) = \frac{\pi}{4} - \beta$$

Then $\sin\beta$ and $\cos\beta$ may be expanded in ascending powers of β, and retaining only the first power the error is less than ·0005.

If then β be assumed equal to ·046298 we find

$$\cos{\cdot}7391 = {\cdot}7390751\ldots,$$

three terms of the expansions giving $\sin\beta + \cos\beta$ to seven decimal places.

Now let

$$\begin{aligned}\cos({\cdot}7391 - \gamma) &= {\cdot}7391 - \gamma,\\ &= {\cdot}7390751 + \delta\sin{\cdot}7391.\\ \therefore \qquad \gamma &= \frac{{\cdot}00002488}{1 + \sin{\cdot}7391},\\ &= \frac{{\cdot}00002488}{1{\cdot}67362},\\ &= {\cdot}0000148659\ldots\end{aligned}$$

and the value of the angle is ·739085134....

Now putting cos·739085134 into the form

$$\frac{\sqrt{2}}{2}\{\cos\cdot046313029+\sin\cdot046313029\}$$

three terms of the expansions give

$$\cos\cdot739085134=\cdot739085132....$$

Thus the real root of the equation is

$$\cdot73908513....$$

correct to eight decimal places.

Next: To find the imaginary roots.

Let $A+B\sqrt{-1}$ be a root, then

$$A+B\sqrt{-1}=\cos A(e^B+e^{-B})/2+\sqrt{-1}\sin A(e^B-e^{-B})/2.$$

$$\therefore\ A=\cos A.(e^B-e^{-B})/2$$

$$\text{and } B=\sin A.(e^B-e^{-B})/2.$$

Now $$1+\{(e^B-e^{-B})/2\}^2=\{(e^B+e^{-B})/2\}^2$$

$$\therefore\ 1+B^2/\sin^2A=A^2/\cos^2A$$

$$\therefore\ B=\pm\tan A\sqrt{A^2\cos^2A,} \qquad \dots \qquad \dots \qquad (2)$$

and $$A=\cos A(e^{\tan A\sqrt{(A^2-\cos^2A)}}+e^{-\tan A\sqrt{(A^2-\cos^2A)}})/2, \qquad (3)$$

Thus A cannot be less than cosA. The values $B=0$, $\cos A=A$, satisfy these equations. This gives the real root already found. But there is another value of A between ·739... and $\pi/2$ which satisfies (3).

For $\cos A\,(e^B+e^{-B})/2=\infty$ when $A=\pi/2$, and equals A when $A=\cdot739...$; and also the rate of change of that expression is less than the rate of change of A, when $A=\cdot739....$ Therefore A and $\cos A.(e^B+e^{-B})/2$ must coincide for another value of A. This value is easily found by trial to be ·9623..... Thus

$$\cdot962...+1\cdot1096\sqrt{-1}$$

is a root of the equation.

And there is no other root between ·962... and $\pi/2$, for the rate of change of $\cos A(e^B+e^{-B})/2$ is greater than the rate of change of A, for these values.

The process of solving equation (3) by trial may be simplified as follows.

There can be no value of A in the second or the third quadrant, which will satisfy the equation, for cosA is negative. Between the values $3\pi/2$ and 2π, of A, the right side of equation (3) changes from ∞ to 0, and it passes through the same values between 2π and $5\pi/2$. Therefore there is a root between the values $A = 2n\pi \pm \pi/2$ and $A = 2n\pi$, where n is any integer.

Let $2n\pi + C$ be a value. Then as a first approximation C may be neglected in comparison with $2n\pi$.

Thus (3) becomes

$$2n\pi = \cos C . e^{2n\pi \tan C}/2.$$

When n is large, C becomes small, and we may put $\tan C = C$ and $\cos C = 1$.

$$\therefore\ C = \log_e 4n\pi/2n\pi, \quad \ldots \quad \ldots \quad \ldots \qquad (4)$$

If a value a, substituted for A in (3), make the right side of the equation $= a + d$, then the correction to be applied to a is very nearly

$$d/4n^2\pi^2, \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (5)$$

For the rate of variation of $e^{n\pi C}/2$ as C changes is $n\pi e^{2n\pi C}$, that is

$$4n^2\pi^2$$

by equation (4).

Now let x be the correction required

$$a - x = (a + d) - 4n^2\pi^2 x.$$

or $$x = d/4n^2\pi^2 \text{ nearly.}$$

By employing these methods the following are the values of the roots obtained,

$$\cdot962 \ldots\ldots\ldots + 1\cdot1096 \ldots \sqrt{-1},$$
$$5\cdot86956 \ldots\ldots + 2\cdot5449 \ldots \sqrt{-1},$$
$$6\cdot6663 \ldots\ldots\ldots + 2\cdot6607 \ldots \sqrt{-1},$$
$$12\cdot30856 \ldots\ldots + 3\cdot2349 \ldots \sqrt{-1},$$
$$12\cdot817498 \ldots\ldots + 3\cdot2799 \ldots \sqrt{-1},$$
$$18\cdot657 \ldots\ldots\ldots + 3\cdot6191 \ldots \sqrt{-1},$$

etc.

For higher values of n, the value of C given by (4) is true to at least two decimal places, and the approximation given by using equation (5) is correct to at least four places.

For example, when $n=4$, equation (4) gives $A=24{\cdot}975\ldots$ the correct value being $24{\cdot}977\ldots$.

The Wallace line and the Wallace point.

By J. S. MACKAY, LL.D.

In what follows I propose to give the history of two theorems and to state some of the consequences that have been developed from them.

The first theorem is :

If a triangle be inscribed in a circle, and from any point in the circumference perpendiculars be drawn to the sides, the feet of these perpendiculars lie in a straight line.

This straight line is sometimes called the pedal line of the triangle, but it is much more frequently named the Simson line, from the belief that Robert Simson of Glasgow was the discoverer of the theorem. This belief is erroneous, for the theorem is not to be found in any of Simson's published works ; I have searched every one of them for it in vain. It may be worth mentioning also that no writer who has used the appellation Simson line has ever given a reference to any passage of Simson's works where the theorem is either stated or implied. How then has this appellation arisen ? The first time that the theorem is attributed to Simson is about 1814 in an article by F. J. Servois in Gergonne's *Annales de Mathématiques*, IV. 250. Servois merely says he believes (*je crois*) the theorem is Simson's. Poncelet in his *Propriétés Projectives*, published in 1822, remarks (§ 468) that Servois attributes the theorem to Simson, and it is, I conjecture, this reproduction of Servois's belief by Poncelet on which succeeding geometers have relied when they bestowed the name Simson line.

If the credit of the discovery of the line may not then be given to Simson, to whom does it belong ? In the *Proceedings of the Edinburgh Mathematical Society*, III. 104 (1885) Dr Thomas Muir mentions the fact that the theorem in question occurs in an article by William Wallace in Leybourn's *Mathematical Repository* (old series), II. 111. Apart from the circumstance that I have not met with the theorem in any writer anterior to Wallace, the following are the reasons which induce me to think that Wallace is the discoverer of the line. Wallace's article in the second volume of the *Mathematical Repository* was preceded by another of his in the first volume (p. 309) in which he states the theorem that if three straight lines touch a parabola, the circle described through their intersections passes through the focus. In his proof of this theorem (II. 54–5) he draws perpendiculars from the focus on the three tangents and shows that the feet of the perpendiculars lie on the tangent at the vertex; in other words, that the tangent at the vertex is the so-called Simson line of the triangle formed by the three other tangents. The supposition here is that the triangle varies according to a certain law, and that the point from which perpendiculars are drawn to its sides is fixed. Change the supposition, and assume that the triangle is fixed and that the point from which perpendiculars are drawn to its sides varies according to a certain law, and the theorem whose origin we are seeking emerges at once. The close connection of the parabola theorem and the so-called Simson line theorem with each other, the latter being readily suggested by the former, and their corresponding proximity in Wallace's articles in the *Repository* lead one to the belief that the so-called Simson line discovery came after the discovery of the parabola property. Poncelet, who doubtless never saw Wallace's articles, treats the two theorems in the same way. See *Propriétés Projectives*, §§ 466–8.

Two pertinent facts remain to be adduced. First, that Simson in his *Sectiones Conicae* does not mention the parabola property; second, that Lambert, who anticipated* Wallace in his discovery of the parabola property, does not give the so-called Simson line property. The discovery of the Wallace line therefore dates back only to about the year 1799 or 1800.

The first generalisation of Wallace's theorem is due to Poncelet,† who states that the perpendiculars on the sides of the triangle may be replaced by obliques making, in cyclical order, equal angles with the sides.

The same generalisation is also made by Steiner in an article ‡ in Gergonne's *Annales*, XIX. 37–64 (1828). Steiner adds that all the straight lines obtained in varying the angle of the obliques will envelope a parabola whose focus is the point from which the obliques are drawn.

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* *Insigniores Orbitae Cometarum Proprietates*, Sectio I. § 15 (1761).

† *Propriétés Projectives*, § 468 (1822).

‡ See Steiner's *Gesammelte Werke*, I. 197.

Poncelet's generalisation is sometimes attributed to Chasles, who gives it in his *Géométrie Supérieure*, § 395, with an interesting proof by involution.

The second generalisation of Wallace's theorem occurs in Gergonne's *Annales*, XIV. 28 (1823) and is probably due to Gergonne himself. It is

If from any point in the circumference of a circle concentric with the circumscribed circle of a triangle perpendiculars be drawn to the sides of the triangle, and their feet be joined, there is formed a triangle of constant area. When this circle becomes the circumscribed circle the area vanishes.

If two circles concentric with the circumscribed circle of a triangle be such that the sum of the squares of their radii is equal to twice the square of the circumscribed radius, and two triangles be formed from them as above, those two triangles are equivalent.

An analytical demonstration is given in the same volume, pp. 280–5, by M. Querret, St. Malo.

The next generalisation of Wallace's theorem is found in *The Mathematician*, I. 70–73 (1845). It is

If from any point in the plane of a triangle perpendiculars be let fall upon the sides, and the extremities of these perpendiculars be joined two and two; if the triangle thus formed be denoted by Δ', the original triangle by Δ, the radius of the circumscribed circle by R, and the distance between the assumed point and the centre of the circumscribed circle by R′, then

$$\frac{\Delta'}{\Delta} = \pm\frac{R^2 - R'^2}{4R^2}$$

according as the assumed point is within or without the circumscribed circle.

When the perpendiculars are replaced by obliques making the same angle ϕ with the sides

$$\frac{\Delta'}{\Delta} = \pm\frac{R^2 - R'^2}{4R^2}\operatorname{cosec}^2\phi$$

The author of this generalisation, whose demonstration is analytical, appends only his initials W. F. [From the list of subscribers to the first volume of *The Mathematician* I conjecture that the name is William Finley] and dates his communication, Oct. 12th, 1843. A geometrical demonstration of the theorem as stated by Finley is given by Mr John Davis, Skibbereen, in *The Mathematician*, II. 37–8 (Nov. 1845). Two others, by Mr J. M'Dowell and Mr Matthew Collins, will be found in *Mathematical Questions with their Solutions from the "Educational Times,"* XVIII. 95–7 (1872). A trigonometrical proof with some applications is given by M. Vuibert in his *Journal de Mathématiques Élémentaires*, IV. 49–52 (1880).

The next generalisation occurs in the *Revue des Sociétés Savantes*, V. 203–233 (1870), where M. Combette has a memoir entitled *Etude d'un lieu géométrique dont il n'a pas été fait mention.* This memoir, the title of which is not quite accurate, and the researches preceding it, establish the following results :

(1) If P be an assumed point, and D, E, F its projections on the sides of triangle ABC, the locus of P when the area of triangle DEF is constant is the circumference of a circle concentric with the circumscribed circle of ABC.

(2) When P coincides with O the circumscribed centre, the area of triangle DEF is one-fourth of the area of ABC.

(3) When OP increases from zero to R, the area of DEF diminishes from one-fourth of ABC to zero, the locus becomes the circumscribed circle itself, and the triangle DEF becomes the Wallace line.

(4) When OP increases from R indefinitely, the area of DEF increases from zero indefinitely.

(5) Hence for every value of the area of DEF included between zero and one-fourth of ABC, the locus of P will consist of two circumferences concentric with the circumscribed circle, the one interior and the other exterior to it.

(6) The sum of the squares of the radii of these two circumferences will be double the square of the circumscribed radius.

(7) The locus of P does not change its nature or its centre when the perpendiculars let fall on the sides become straight lines all making equal angles with the sides.

(8) The relation between the radius R' of the circumference which corresponds to a given value Δ' of the variable area, the radius R of the circumscribed circle, the surface Δ of the given triangle,

and the angle ϕ which the lines drawn from a point of the locus make with the sides is

$$\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{4R^2 \sin^2\phi}.$$

(9) If the triangle ABC is replaced by a plane polygon, and the projections on its sides of a point P in the plane are joined, the locus of P when this area is constant is still a circle, which has always the same centre whatever be the value of the area.

(10) When the point P is taken in space and projected on the sides of a plane polygon, the locus of P when the area obtained by joining the projections is constant, becomes a cylindrical surface of revolution whose axis perpendicular to the plane of the polygon is always the same for all values of the area.

A large number of the theorems connected with the Wallace line will be found in an article by Mr John Alison in the *Proceedings of the Edinburgh Mathematical Society*, III. 77–93 (1885).

The second theorem is

If four straight lines intersect each other and form four triangles, circles circumscribed about each of these triangles will pass through one and the same point.

This theorem was proposed for proof in 1804 in the first volume of Leybourn's *Mathematical Repository* (new series), p. 22 (pagination of the Questions) by Scoticus. In reference to Scoticus, I make the following quotation from an article by Mr T. T. Wilkinson in the *Mechanics' Magazine*, Vol. LV., p. 447 (1851). "This fictitious signature is one of those adopted by Dr Wallace, as appears from a manuscript note in the late Professor Davies's copy of the *Mathematical Repository*, on the authority of Professor Leybourn, the editor."

Two solutions of the theorem are given in the *Repository*, Vol. I., Part I., p. 170, but they are not essentially different. No additional properties connected with this figure, which is that of a complete quadrilateral, seem to have appeared in the *Repository* till about 1832, when Professor Davies pointed out four (see Vol. VI., Part I., pp. 124–7, and Question 555). One of these he had proposed in the *Leeds Correspondent* in 1821; another, which forms question 555, is the second of Steiner's series of theorems.

In Gergonne's *Annales*, XVIII. 302 (1828) Steiner enunciates without proof the following theorems regarding four intersecting straight lines :

1°. These four straight lines, taken three by three, form four triangles such that their circumscribed circles pass all through the same point P.

2°. The centres of these four circles and the point P are on the circumference of a fifth circle.

3°. The feet of the perpendiculars* let fall from the point P on the four straight lines are situated on one straight line R, and this property belongs exclusively to the point P.

4°. The orthocentres of the four triangles (1°) are situated on one straight line R′.

5°. The straight lines R and R′ are parallel, and the straight line R passes through the middle of the perpendicular let fall from P on R′.

6°. The middle points of the diagonals of the complete quadrilateral formed by the four straight lines (1°) are situated on one straight line† R″.

7°. The straight line R″ is perpendicular to the two straight lines R, R′.

8°. For each of the four triangles (1°) there are an inscribed circle and three escribed circles, making in all sixteen circles, whose centres are four by four on one circumference so as to give rise to eight new circles.

9°. These eight new circles are divided into two groups such that each of the four circles of one of these groups cuts orthogonally all the circles of the other group ; whence it follows that the centres of the circles of the two groups are on two straight lines perpendicular to each other.

10°. Finally, these last two straight lines intersect at the point P mentioned above.

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* This property is extended to equally inclined obliques by Mr J. W. Elliott in the *Lady's and Gentleman's Diary* for 1852, p. 71.

† Steiner attributes this theorem to Newton, but gives no reference to the passage where it occurs. It is usually ascribed to Gauss under the date 1810 (Zach, *monatl. Korresp.* 22 p. 115) ; but it was given by Mr J. T. Connor, Lewes Academy, in the *Ladies' Diary* for 1795.

Professor W. K. Clifford, in discussing[*] Wallace's theorem states that it is the third of a series.

"If we take any two straight lines they determine a point, viz., their point of intersection.

"If we take three straight lines we get three such points of intersection; and these three determine a circle, viz., the circle circumscribing the triangle formed by the three lines.

"Four straight lines determine four sets of three lines by leaving out each in turn; and the four circles belonging to these sets of three meet in a point.

"In the same way five lines determine five sets of four, and each of these sets of four gives rise, by the proposition just proved, to a point. It has been shown by Miquel[†] that these five points lie on the same circle.

"And this series of theorems has been shown[‡] to be endless. Six straight lines determine six sets of five by leaving them out one by one. Each set of five has, by Miquel's theorem, a circle belonging to it. These six circles meet in the same point, and so on for ever. Any even number ($2n$) of straight lines determines a point as the intersection of the same number of circles. If we take one line more, this odd number ($2n+1$) determines as many sets of $2n$ lines, and to each of these sets belongs a point; these $2n+1$ points lie on a circle."

In connection with Wallace's two theorems, mention may be made of the principle of what M. De Longchamps calls Recurrent Geometry,[§] and of his application of it.

Imagine three straight lines d_1, d_2, d_3 in a plane. Let P be a remarkable and well defined point of this triangle; for example, the centroid, the orthocentre, the circumscribed centre, and so on. To mark that P is a well determined point of the triangle $d_1\ d_2\ d_3$ let it be denoted by P_{123}.

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[*] Clifford's *Common Sense of the Exact Sciences*, pp. 30-1 (1885).

[†] Liouville's *Journal*, III. 486-7 (1838).

[‡] Clifford in the *Oxford, Cambridge, and Dublin Messenger of Mathematics*, V. 124-141 (1871).

[§] See *Journal de Mathématiques Élémentaires* (2nd series), II. 3-10, 25-33, 49-56, 73-8, 121-6 (1883). The substance of these papers had appeared in 1877 in the *Nouvelle Correspondance Mathématique*, III. 306-312, 340-7, and in the *Annales scientifiques de l'Ecole normale*, III. 321-341 (1866).

Now take four straight lines d_1, d_2, d_3, d_4. If one of them, d_4 for example, be removed, there remains a triangle $d_1d_2d_3$ to which corresponds the point P_{123} previously defined, and on which it is proposed to make a study of recurrent geometry. If one of the four given straight lines be successively removed, four points are obtained

$$P_{123}, P_{234}, P_{341}, P_{412}.$$

These four points possess, whether from their respective positions with regard to each other, or from their situation with regard to the given straight lines, geometrical properties which may in certain cases be remarkable. From a geometrical study of them there may result the discovery of a point which will be denoted by P_{1234}. This point may possess with regard to the straight lines d_1, d_2, d_3, d_4, or with regard to the points P_{123}, P_{234}, P_{341}, P_{412} geometrical properties which recall those which connected the point P_{123} to the triangle $d_1d_2d_3$.

Let the point P_{123}, defined by the triangle, be called a P point of the 3rd order, and the analogous point P_{1234}, which corresponds to four straight lines, be called a P point of the 4th order.

Now take five straight lines, and successively remove one of them, d_5 for instance. We shall have to consider a complete quadrilateral formed by the four other straight lines, and to this quadrilateral corresponds a P point of the 4th order, which will be denoted by P_{1234}. Thus five points of the 4th order are obtained

$$P_{1234}, P_{2345}, P_{3451}, P_{4512}, P_{5123}.$$

If with these five points there can be obtained a P point which is deduced from them as the P point of the 4th order was deduced from the P points of the 3rd order, this point will be a P point of the 5th order, and will be denoted by P_{12345}.

M. De Longchamps has applied his principle of recurrent geometry to the circumscribed centre, to the centroid, to the nine point circle, and, as follows, to the Wallace line.

If from a point P (which will be called the fundamental point) taken on a circle O, perpendiculars be drawn to the sides of a triangle whose vertices 1, 2, 3 are situated on O, the feet of these perpendiculars are in one straight line d_{123}.

Take now four points 1, 2, 3, 4 on the circle O. If one of these points be successively removed, four triangles will be obtained, and to each of them along with the point P will correspond a Wallace line of the 3rd order,

$$d_{123},\ d_{234},\ d_{341},\ d_{412}.$$

If from the fundamental point P perpendiculars be drawn to these four Wallace lines of the 3rd order, the four points thus obtained are situated on a straight line. This straight line will be a Wallace line of the 4th order, and will be denoted by d_{1234}.

Similarly if five points 1, 2, 3, 4, 5 be taken on a circle O, the projections of the fundamental point P on the five Wallace lines of the 4th order will be situated on a straight line. This straight line will be a Wallace line of the 5th order, and will be denoted by d_{12345}.

Hence the following general theorem :

If n points be taken on a circle, and one of them be removed, $(n-1)$ points are obtained to which correspond a Wallace line of the $(n-1)^{th}$ order. Thus there are n Wallace lines of the $(n-1)^{th}$ order. The projections of the fundamental point P on these n lines are situated on one straight line, called a Wallace line of the n^{th} order, and denoted by $d_{12 \ldots n}$.

Note on an Equation of Motion.

By A. J. Pressland, M.A.

It can be shown by means of relative motion that if two bodies A and B move with velocities u and v in the same straight line, and a third body C move with velocity $u+v$ also in the same straight line, the space passed over by C is equal to the sum of the spaces passed over by A and by B in the same time.

Let A move with an initial velocity u and an acceleration f for an interval t.

Its velocity at the end of the interval will be $u+ft$ which call v.

Then $$u+ft=v \text{ or } u=v-ft.$$

Now let B move with velocity v and acceleration $-f$. Its velocity at the end of the interval t will be $v-ft$, that is u.

Hence the motion of B is the exact counterpart, or reverse, of that of A. Therefore each passes over the same space s.

Hence C passes over a space $2s$.

But C moves with uniform velocity $u+v$, for the increase of velocity in A is neutralised by a corresponding decrease in B.

Therefore C passes over a space $(u+v)t$.

Hence $$2s=(u+v)t$$
$$=(2u+ft)t$$
or $$s=ut+\tfrac{1}{2}ft^2.$$

Therefore a body moving with initial velocity u and an acceleration f passes over in time t a space denoted by $ut+\frac{1}{2}ft^2$.

Edinburgh Mathematical Society.

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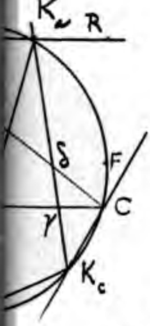

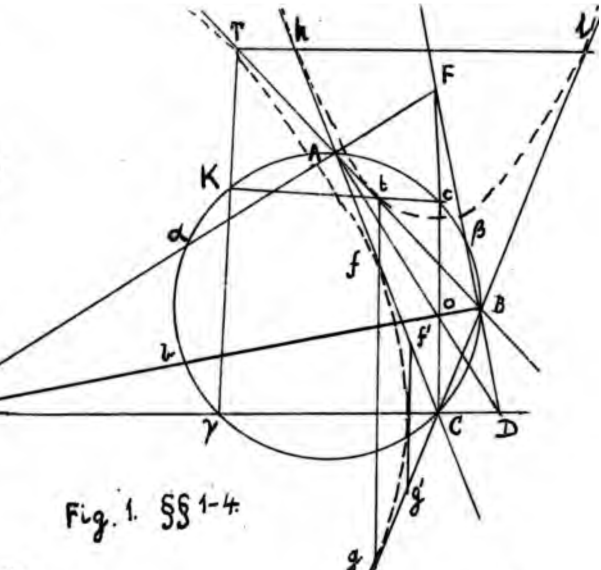

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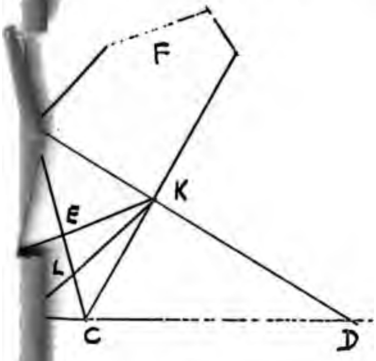

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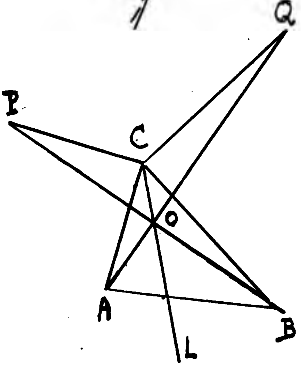

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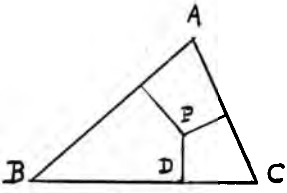

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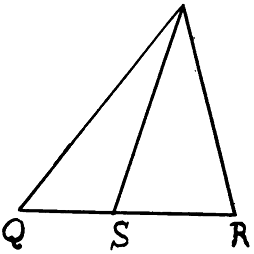

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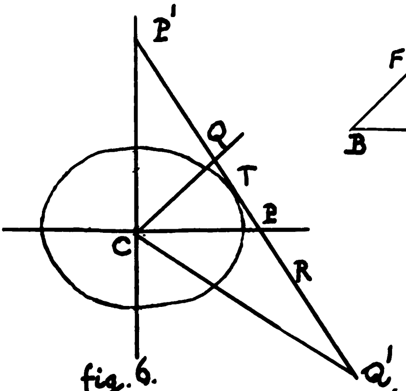

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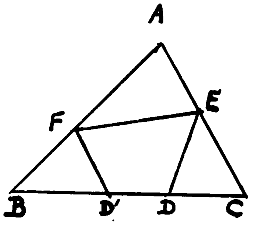

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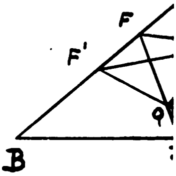

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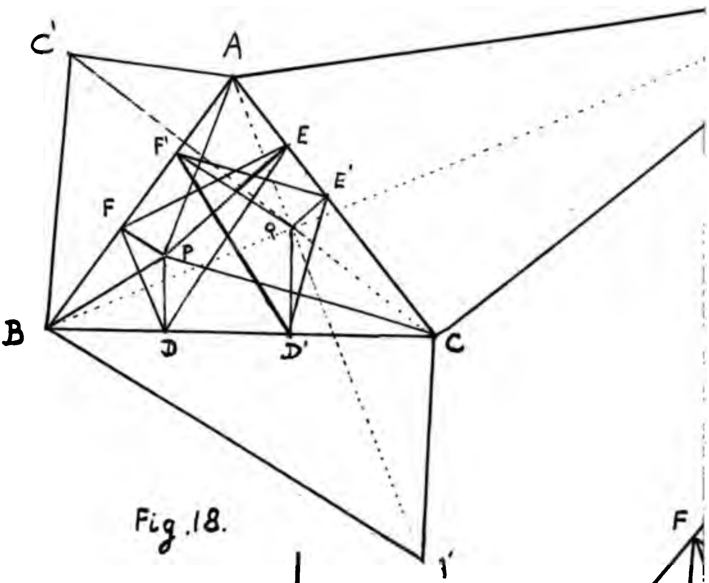

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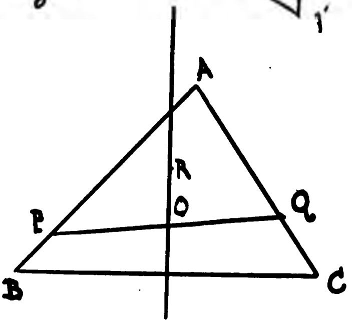

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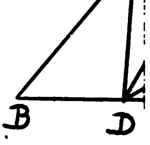

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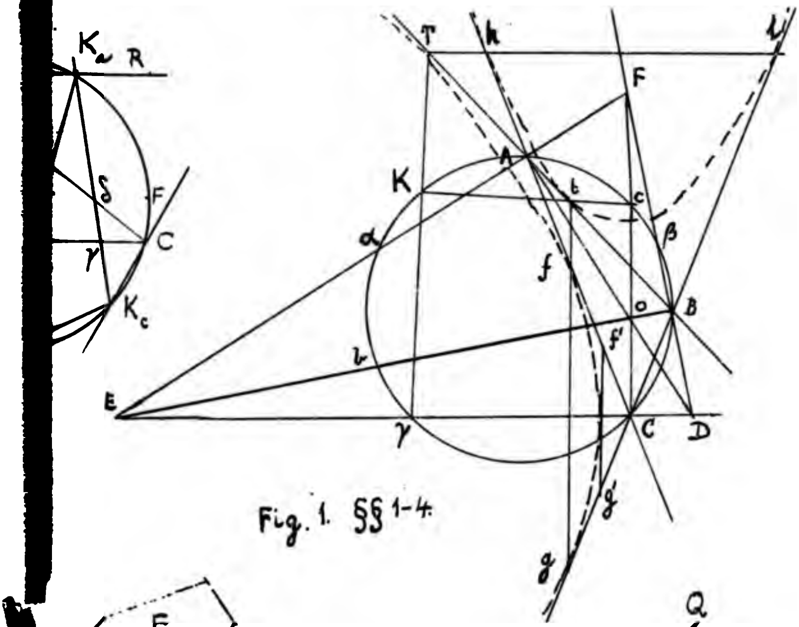

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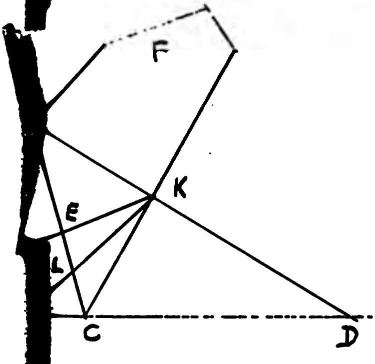

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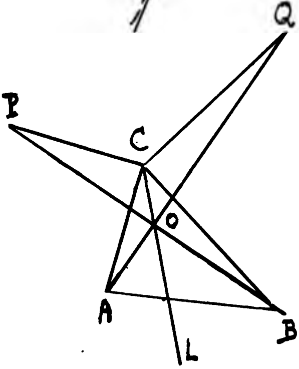

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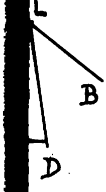

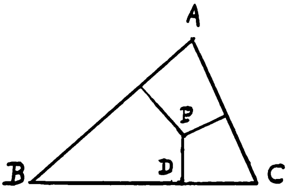

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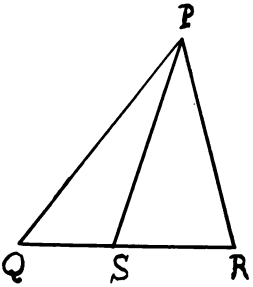

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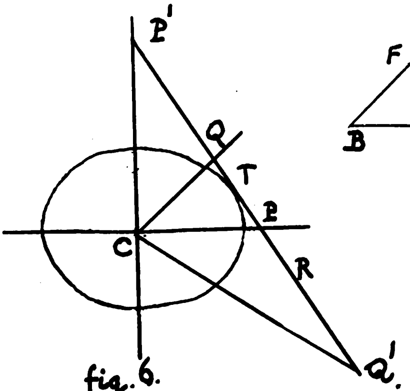

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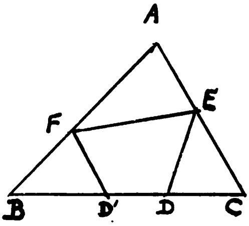

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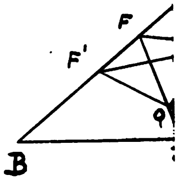

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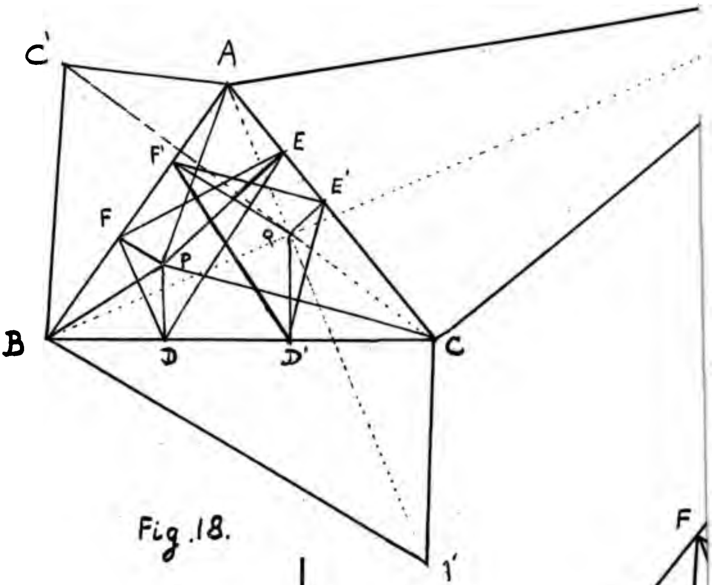

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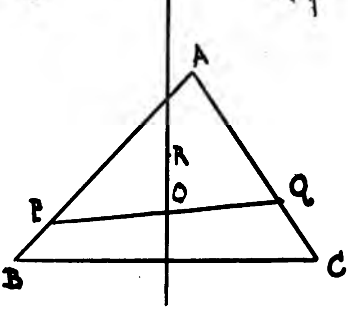

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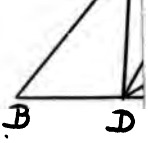

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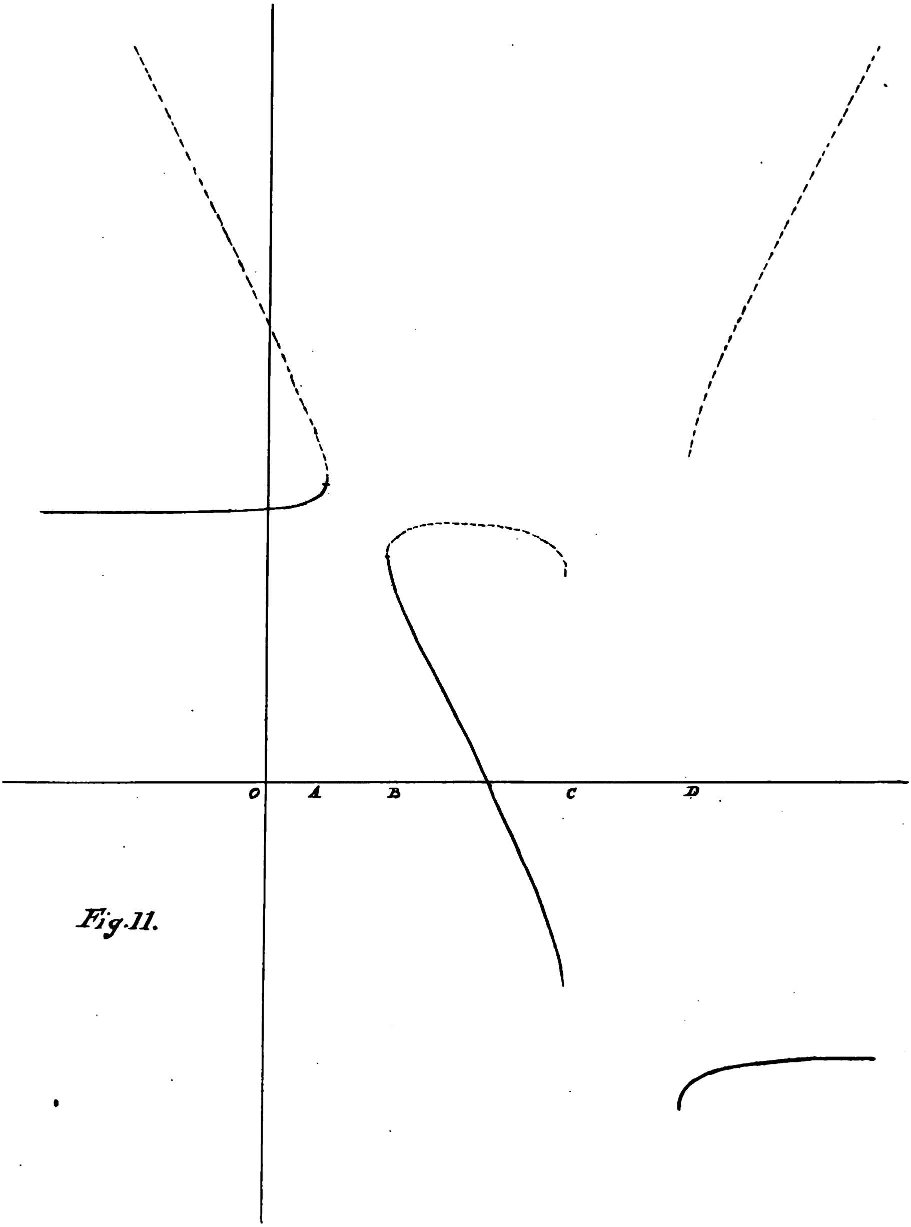

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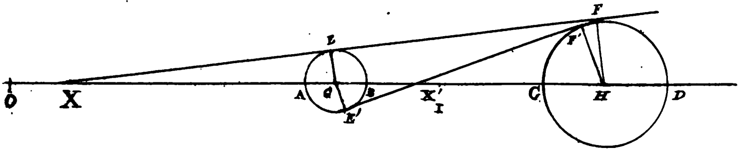

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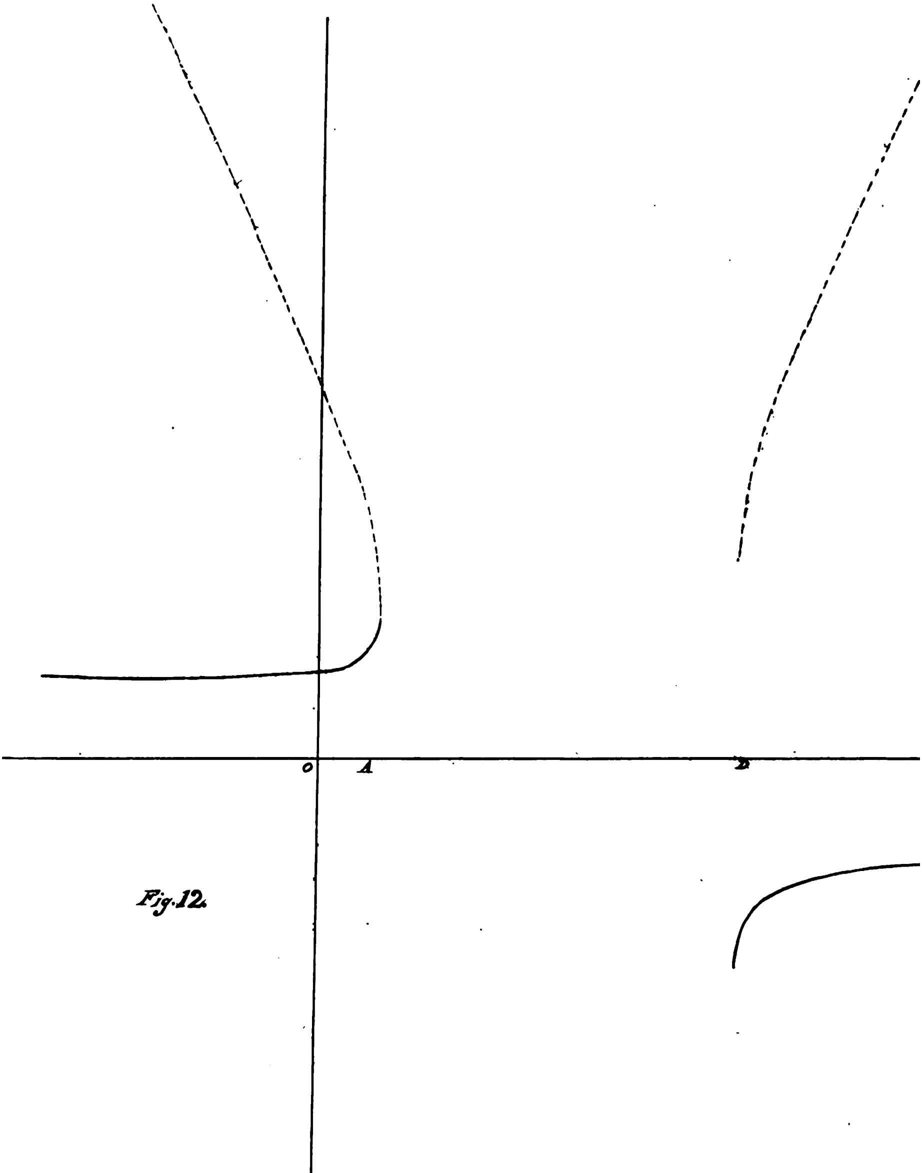

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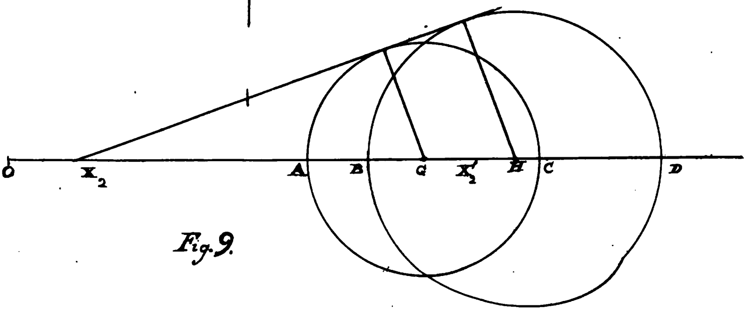

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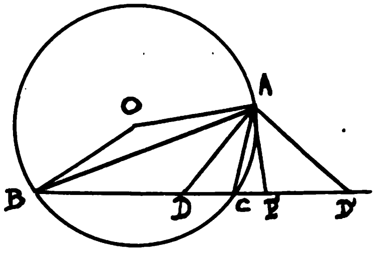

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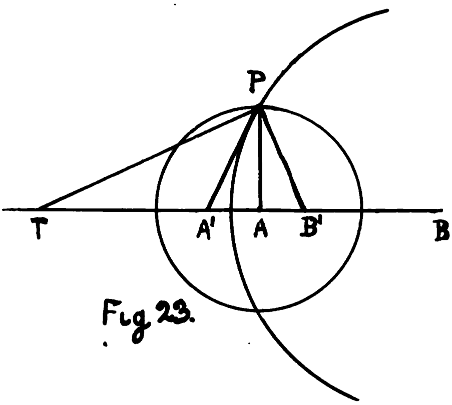

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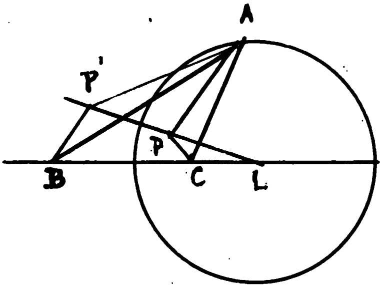

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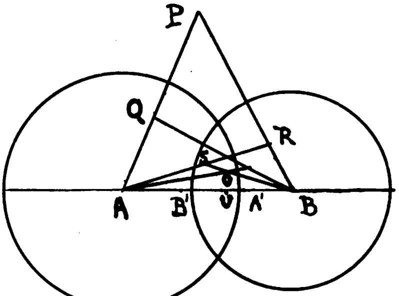

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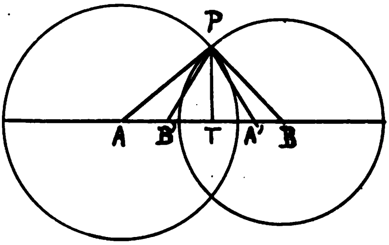

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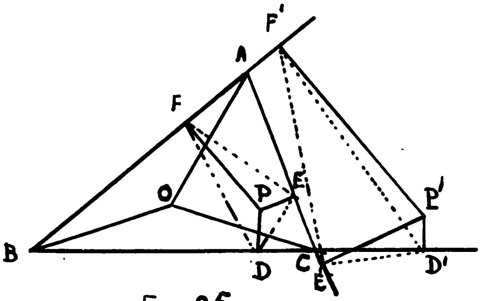

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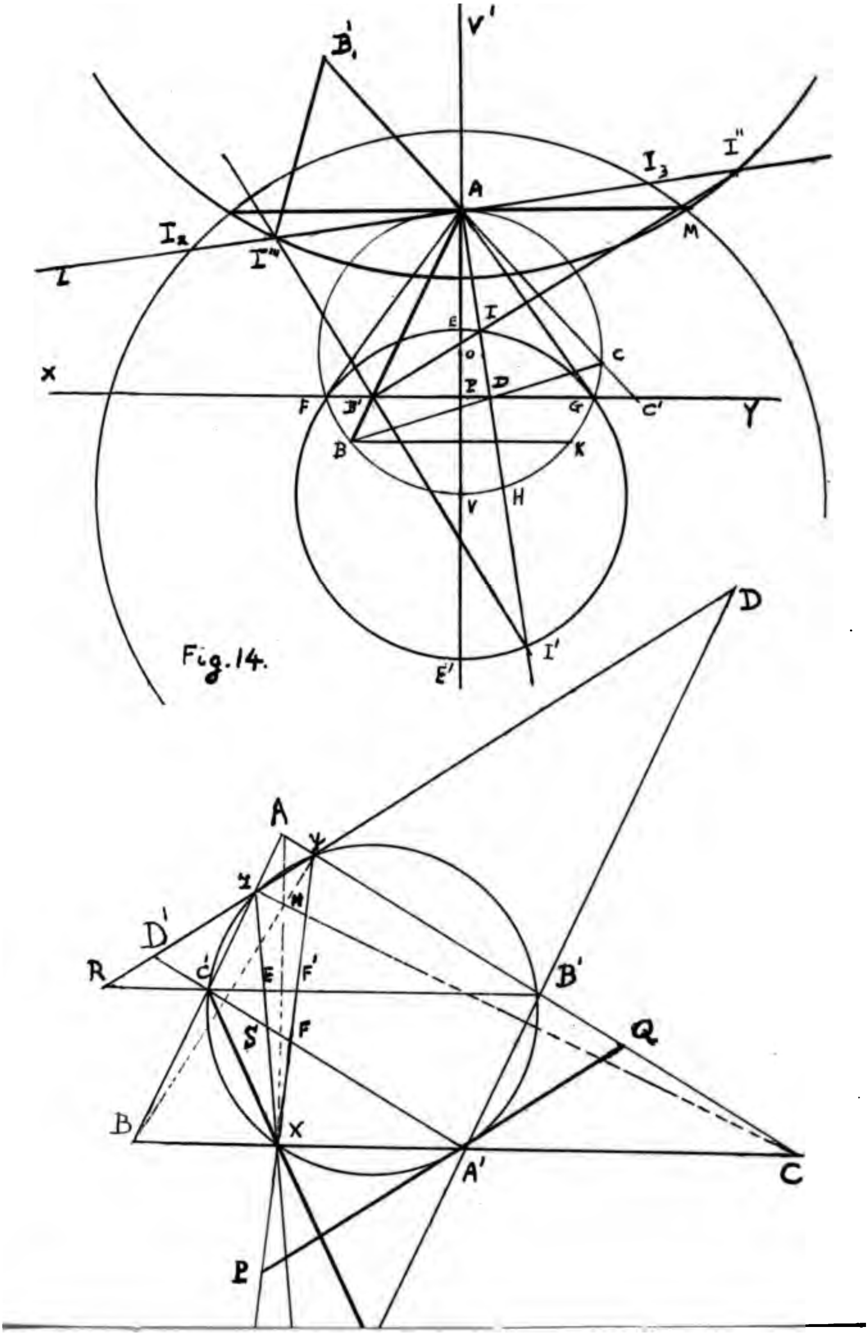

Fig. 14.

PROCEEDINGS

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VOLUME X.

SESSION 1891–92.

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OF THE

EDINBURGH OBSTETRICAL SOCIETY.

[illegible]

First Meeting, November [illegible], 1891.

[illegible] F.R.C.P.E., in the Chair.

For the Session the following Office-bearers were elected.—

President—[illegible], F.R.C.P.E.

Vice-Presidents—[illegible], F.R.C.P.E.

Secretary—[illegible], F.R.C.P.E.

Treasurer—[illegible], F.R.C.P.E.

Editor of Transactions—[illegible], F.R.C.P.E.

Council.

[illegible], F.R.C.P.E., [illegible]

[illegible]

The Barycentric Calculus of Möbius.

By R. E. Allardice, M.A.

August Ferdinand Möbius was born at Schulpforta, in Saxony, in the year 1790. He studied in the Universities of Leipsic and Göttingen, and, at the age of 26, was appointed extraordinary Professor of Astronomy and Superintendent of the Observatory at Leipsic. There he remained till his death, in 1868, being appointed ordinary professor in 1844. Between the years 1817 and 1868 Möbius wrote his Barycentric Calculus, a Treatise on Statics, another on the Mechanics of the Heavens, and a large number of papers on Mathematical, Dynamical, and Astronomical questions. Most of these papers were contributed to Crelle's Journal, which was founded in 1826. The works of Möbius have recently been collected under the direction of the Royal Scientific Society of Leipsic, and under the editorship of Klein, Scheibner, and Baltzer.

The Barycentric Calculus was published in 1827, and forms nearly two-thirds of the first volume of the collected works of Möbius. Though this Calculus is thus nearly two-thirds of a century old, its methods have never been introduced into ordinary mathematical courses, and I have, therefore, thought it worth while to give some account of it. The exposition of his Calculus, as given by Möbius, is exceedingly clear, and shows that, although the methods of the Calculus have not so wide an application as those of ordinary Co-ordinate Geometry, yet, within their own range, they are very powerful in the hands of one who has thoroughly mastered them. The work also possesses a large amount of historical interest. It was here that homogeneous co-ordinates were first used—Plücker's *Analytisch-geometrishe Entwickelungen*, in which the theory of trilinear co-ordinates, as they are now used, was first given, not appearing till 1828 (the second volume not till 1831.) Here also Möbius gave, for the first time, the method of determining the sign of a segment of a line in accordance with the, arbitrarily assumed, positive direction of the line (what may be called the quaternion method of determining the sign of a segment of a line); and discussed also, for the first time, the sign of an area and the sign of a volume. The distinction should be noted between the methods used for determining the sign of a line in quaternions and in co-ordinate geometry. Though one of these methods may be deduced from the other, they are really fundamentally different.

Some slight idea of the method of the Calculus may be given in a few words as follows :—

Let us confine our attention to a plane, and consider three points, A, B, C. If weights a, b, c, be imagined at the points A, B, C, then this system will have a definite centre of gravity, P say. In the determination of P only the ratios $a:b:c$ are involved ; and there is a one-one correspondence between the position of P and the values of these ratios. These quantities a, b, c, or their ratios, are therefore taken as the co-ordinates of P ; but, as will be seen later, the whole theory is established on a purely geometrical basis. It is obvious that this method may be extended to space of three dimensions, by taking four non-coplanar points. In the development of his theory, Möbius always represents a curve or surface, not by means of an equation connecting the co-ordinates, but by expressing the co-ordinates as functions of one or two independent variables.

Möbius begins by pointing out how signs are to be attached to segments of lines in accordance with the definition $\overline{\mathrm{AB}}+\overline{\mathrm{BA}}=0$.

He then proceeds to give a purely geometrical theory of the centre of gravity, somewhat as follows :—

Through A and B (fig. 1) a pair of parallel straight lines are drawn ; to draw A′B′ so that $a.\mathrm{AA}'+b.\mathrm{BB}'=0$.

Any straight line through a definite point P satisfies this condition.

P is the centre of gravity of the points A and B with weights a and b ; and is between a and b if a and b are both positive.

If A″P″B″ is any line, not through P, then

$$a.\mathrm{AA}''+b.\mathrm{BB}''=(a+b)\mathrm{PP}''.$$

It is shown next that if A, B, C, (fig. 2) are three given points, then a point Q may always be found such that

$$a.\mathrm{AA}'+b.\mathrm{BB}'+c.\mathrm{CC}'=0$$

$$\text{and } a.\mathrm{AA}''+b.\mathrm{BB}''+c.\mathrm{CC}''=(a+b+c).\mathrm{QQ}''.$$

Q is obviously the C.G. of the system A, B, C, with the weights a, b, c.

A similar theorem is next proved for four points, whether these be coplanar or not.

Any number of fundamental points A, B, C, ... N may be employed. A definite set of co-efficients (a, b, c...n) (or weights) determine one definite point (the centre of gravity).

Particular case.

If $a+b+c\ldots+n=0$, the point is at infinity.

Since $a \neq 0$, $\therefore\ b+c+\ldots n \neq 0$.

We may therefore determine T so that, if lines be drawn through B, C,...T parallel to a given line and be cut by any plane in B′,C′,...T′, then

$$b.\mathrm{BB}'+c.\mathrm{CC}'+\ldots+n.\mathrm{NN}'=(b+c\ldots+n).\mathrm{TT}'=-a.\mathrm{TT}'$$

$$\therefore\ a.\mathrm{AA}'+b.\mathrm{BB}'+\ldots.+n.\mathrm{NN}'=a.\mathrm{AA}'-a.\mathrm{TT}'.$$

Now this will be zero, provided the plane, B′C′...T′ be parallel to AT; *i.e.*, pass through the point at infinity in AT. Hence this point at infinity is the point required.

If then three fixed points A, B, C, be given in a plane, a point P is determined in accordance with the above when the coefficients a, b, c, are given.

We have then the relation

$$a.\mathrm{AA}'+b.\mathrm{BB}'+c.\mathrm{CC}'=(a+b+c).\mathrm{PP}'$$

This is written more shortly

$$a\mathrm{A}+b\mathrm{B}+c\mathrm{C}=(a+b+c).\mathrm{P}$$

And the notation $a\mathrm{A}+b\mathrm{B}+c\mathrm{C}\equiv\mathrm{P}$ is also employed to denote that P is the point whose co-ordinates are a, b, c.

Möbius next considers the application of algebraical signs to areas and to volumes; and then establishes the following results:—

If we put $c=-(a+b)$, we may denote the same fact in any of the three ways:—

$$a\mathrm{A}+b\mathrm{B}\equiv\mathrm{C},\ a\mathrm{A}+b\mathrm{B}=(a+b)\mathrm{C},\ a\mathrm{A}+b\mathrm{B}+c\mathrm{C}=0.$$

In this case, $a:b:c=\mathrm{BC}:\mathrm{CA}:\mathrm{AB}$; and A, B, C, are collinear.

If $a\mathrm{A}+b\mathrm{B}+c\mathrm{C}\equiv\mathrm{D}$, then A, B, C, D, are coplanar and $a:b:c=\triangle\mathrm{DBC}:\triangle\mathrm{DCA}:\triangle\mathrm{DAB}$.

If A, B, C, D are not coplanar, and if $a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+d\mathrm{D}\equiv\mathrm{E}$,

$$\text{then } a:b:c:d=\mathrm{BCDE}:\mathrm{CDEA}:\mathrm{DEAB}:\mathrm{EABC}$$

where BCDE, etc., denote volumes of pyramids.

It is next shown that if A, B, C are any three non-collinear points in a plane, the ratios $a:b:c$ may be so determined that $a\mathrm{A}+b\mathrm{B}+c\mathrm{C}\equiv\mathrm{D}$, where D is any fourth point whatever; that is, any point in a plane may be regarded as the C.G. of any other three points.

Let AB and CD intersect in L; then we may write

$\mathrm{L}\equiv\alpha\mathrm{A}+\beta\mathrm{B}\equiv\gamma\mathrm{C}+\delta\mathrm{D}$, where the ratios $\alpha:\beta$ and $\gamma:\delta$

are determined. Now determine the ratio $\beta : \gamma$ so that $\alpha + \beta = \gamma + \delta$, then $\alpha A + \beta B = \gamma C + \delta D$

$$\therefore \; D \equiv \alpha A + \beta B - \gamma C.$$

A similar proposition is proved for any four non-coplanar points in space and any fifth point taken arbitrarily.

Taking now $pA + qB + rC \equiv P$ as the expression for a point in the plane, we can determine the region in which P lies according to the relative signs and magnitudes of p, q, r. The result is as follows (fig. 3):—

P lies

$$\begin{array}{ll} \text{I.} & \text{in } \overline{ABC}, \text{ if } p, q, r \text{ have the same sign} \\ \text{II.} & \left\{\begin{array}{l} \text{in } \overline{BC}, \text{ if } q+r>p \\ \text{in } \overline{CA}, \text{ if } r+p>q \\ \text{in } \overline{AB}, \text{ if } p+q>r \end{array}\right. \\ \text{III.} & \left\{\begin{array}{l} \text{in } \overline{A}, \text{ if } p>q+r \\ \text{in } \overline{B}, \text{ if } q>r+p \\ \text{in } \overline{C}, \text{ if } r>p+q \end{array}\right. \end{array}$$

it being understood that letters on the same side of the symbol $>$ have the same sign, those on different sides opposite signs.

A similar investigation is then given for space of three dimensions.

Change of Triangle of Reference.

The method of effecting a change of the triangle of reference will be most easily explained by means of the following simple example given by Möbius.

Let A′B′C′ be the new triangle of reference, where A′ is the mid-point of BC, etc.

Then

$$2A' = B + C, \; 2B' = C + A, \; 2C' = A + B, \;\; \therefore A = B' + C' - A', \text{ etc.}$$

If $P \equiv pA + qB + rC$ is any point referred to ABC, its expression, when referred to A′B′C′, will be

$$\begin{aligned} P &\equiv p(B' + C' - A') + q(C' + A' - B') + r(A' + B' - C') \\ &\equiv (q + r - p)A' + (r + p - q)B' + (p + q - r)C'. \end{aligned}$$

Equations to Straight Lines and Planes.

If E and E′ are any two points and w be taken as variable, $E + wE'$ may represent any point on the line joining E and E′ and may therefore be taken as the expression for the straight line EE′.

If A, B, C, are any three non-collinear points in the plane, we may put $e\mathrm{E} = a\mathrm{A} + b\mathrm{B} + c\mathrm{C}$, $e'\mathrm{E}' = a'\mathrm{A} + b'\mathrm{B} + c'\mathrm{C}$; $\therefore$ putting $w = ve'/e$,

$$\mathrm{E} + w\mathrm{E}' \equiv e\mathrm{E} + ve'\mathrm{E}' = a\mathrm{A} + b\mathrm{B} + c\mathrm{C} + v(a'\mathrm{A} + b'\mathrm{B} + c'\mathrm{C})$$
$$= (a + va')\mathrm{A} + (b + vb')\mathrm{B} + (c + vc')\mathrm{C},$$

which is therefore the expression for the line joining the two points E and E′.

A simpler form may be got as follows:—
$\mathrm{P} \equiv \mathrm{A} + b\mathrm{C}$ and $\mathrm{Q} \equiv \mathrm{B} + c\mathrm{C}$ are any points in AC and BC respectively. The expression for PQ is $\mathrm{A} + b\mathrm{C} + x(\mathrm{B} + c\mathrm{C})$, that is, $\mathrm{A} + x\mathrm{B} + (b + cx)\mathrm{C}$.

Intersection of Two Straight Lines.

To determine the intersection of

$$\left.\begin{array}{r}\mathrm{A} + x\mathrm{B} + (a + bx)\mathrm{C} \\ \text{and } \mathrm{A} + y\mathrm{B} + (a' + b'y)\mathrm{C}\end{array}\right\} \text{ we have the relations}$$

$$1 : x : (a + bx) = 1 : y : (a' + b'y)$$
$$\therefore x = y = -(a - a')/(b - b');$$

and the point of intersection is $(b - b')\mathrm{A} - (a - a')\mathrm{B} + (a'b - ab')\mathrm{C}$.

Straight Line in Space.

It may be shown very easily that a straight line in space may be written in either of the forms

$$(a + a'v)\mathrm{A} + (b + b'v)\mathrm{B} + (c + c'v)\mathrm{C} + (d + d'v)\mathrm{D} \text{ or}$$
$$\mathrm{A} + x\mathrm{B} + (a + bx)\mathrm{C} + (c + dx)\mathrm{D}.$$

Plane.

If E, F, G, are three non-collinear points, any point in the plane EFG may be represented by $\mathrm{E} + v\mathrm{F} + w\mathrm{G}$, which is therefore the expression for the plane, when v and w are taken as variables.

Hence any plane may be written

$$(a + a'v + a''w)\mathrm{A} + (b + b'v + b''w)\mathrm{B} + \text{etc.} + \text{etc.}$$

This plane is determined by the points (a, b, c, d), (a', b', c', d') (a'', b'', c'', d''). [*N.B.*—Möbius does not use this notation.]

By taking for the points that determine the plane, points in three of the fundamental lines, we may reduce the expression to the form $\mathrm{A} + x\mathrm{B} + y\mathrm{C} + (a + bx + cy)\mathrm{D}$.

Möbius works out the expression for the line of intersection of two planes, the condition that two planes be parallel, and a number of analagous problems. As a single example, I shall take the following:—

Given the line $(au+a')\mathrm{A}+(\beta u+\beta')\mathrm{B}+u\mathrm{C}+\mathrm{D}$,
and the plane $\mathrm{A}+v\mathrm{B}+w\mathrm{C}+(a+bv+cw)\mathrm{D}$,
to find (1) their point of intersection, (2) the condition that the plane contain the line, (3) the condition that they be parallel.

For every point on their intersection we must have

$$au+a':\beta u+\beta':u:1=1:v:w:a+bv+cw \quad \ldots \quad \ldots \quad (1).$$

These three equations determine u, v and w, giving

$$u=(1-a\alpha'-b\beta')/(a\alpha+b\beta+c).$$

We get the point of intersection by substituting this value in the expression for the given straight line. Putting $m=1-a\alpha'-b\beta'$, $n=a\alpha+b\beta+c$, the point is $(\alpha m+\alpha' n)\mathrm{A}+(\beta m+\beta' n)\mathrm{B}+m\mathrm{C}+n\mathrm{D}$.

If the plane contain the straight line, the equations (1) must hold good for all values of u; hence $m=n=0$.

If the plane and the straight line are parallel, then

$$(\alpha m+\alpha' n)+(\beta m+\beta' n)+m+n=0,$$
$$\therefore\ (a\alpha'+b\beta'-1)/(a\alpha+b\beta+c)=(\alpha+\beta+1)/(\alpha'+\beta'+1).$$

Möbius next proceeds to the discussion of curved lines and surfaces. A curved line in a plane may be represented by the expression $p\mathrm{A}+q\mathrm{B}+r\mathrm{C}$ where p, q, r are given functions of a single variable.

A curve of the second degree may be represented by

$$(a+a'v+a''v^2)\mathrm{A}+(b+b'v+b''v^2)\mathrm{B}+(c+c'v+c''v^2)\mathrm{C}$$

provided the three coefficients do not contain a common factor and provided they cannot all by some transformation be reduced to linear form. Thus we must not have $a=b=c=0$, or $a'=b'=c'=0$. Each side of the triangle of reference is cut in two points, real or imaginary; and so is every other line, as it may be made a side of a new triangle of reference.

If we put

$$a_1\mathrm{A}_1=a\mathrm{A}+b\mathrm{B}+c\mathrm{C}$$
$$b_1\mathrm{B}_1=a'\mathrm{A}+b'\mathrm{B}+c'\mathrm{C}$$
$$c_1\mathrm{C}_1=a''\mathrm{A}+b''\mathrm{B}+c''\mathrm{C}$$

and take $\mathrm{A}_1\mathrm{B}_1\mathrm{C}_1$ as the new triangle of reference, the expression becomes

$$a_1\mathrm{A}_1+b_1v\mathrm{B}+c_1v^2\mathrm{C}_1$$

or when we put $c_1v/b_1=w$, and $a_1c_1/b^2=a$,

$$a\mathrm{A}_1+w\mathrm{B}_1+w^2\mathrm{C}_1 \quad \ldots \quad \ldots \quad \ldots \quad (1).$$

To find the points at infinity on this curve we have to put $a+w+w^2=0$. The points will be real and different, imaginary, or equal, *i.e.*, the curve a hyperbola, ellipse or parabola, according as $a < > = 1/4$.

For these different cases the expression (1) may be reduced to one or other of the forms

$$A + xB + x^2C, \quad -A + xB + x^2C, \quad \tfrac{1}{4}A + xB + x^2C.$$

These curves pass through A and C and touch BA and BC at these points.

Other particular forms of the equation.

$$a(v-\beta)(v-\gamma)A + b(v-\gamma)(v-a)B + c(v-a)(v-\beta)C$$

represents a conic passing through A, B and C.

$$a(v-a)^2A + b(v-\beta)^2B + c(v-\gamma)^2C$$

represents a conic inscribed in ABC.

The next point discussed is the number of independent constants involved in the expression $pA + qB + rC$ where p, q, and r are integral functions of a single variable of the n^{th} degree. The number is $3n-1$, which is less by $\frac{1}{2}(n-1)(n-2)$ than the number involved in the general equation of the n^{th} degree in Cartesian Co-ordinates. As a matter of fact, when p, q, and r are rational functions, the above expression can only represent unicursal curves.

Möbius next considers the question of the contact of curves and of cusps and points of inflexion. The following will give some idea of his method.

Let $pA + qB + rC$ denote the curve, where p, q and r are functions of v. Consider a particular value v' of v and an adjacent value $v' + x$, then

$$p = p' + \frac{dp'}{dv'} \cdot x + \frac{1}{2!}\frac{d^2p'}{dv'^2} \cdot x^2 + \ldots$$

$$q = q' + \frac{dq'}{dv'} \cdot x + \frac{1}{2!}\frac{d^2q'}{dv'^2} \cdot x^2 + \ldots, \text{ etc.}$$

Now put $p'A + q'B + r'C = a_1A_1$,

$$\frac{dp'}{dv'}A + \frac{dq'}{dv'}B + \frac{dr'}{dv'}C = b_1B_1, \text{ etc.}$$

Then the point on the curve corresponding to $v' + x$, is

$$a_1A_1 + b_1xB_1 + c_1x^2C_1 + \ldots\ldots$$

This, as a matter of fact, is another form of the expression for the curve, the new variable x occurring in the n^{th} degree. The first two terms $a_1A_1 + xb_1B_1$ give the tangent at the point (p', q', r'). Möbius examines thoroughly the nature of the contact of the curves of the 2^{nd}, 3^{rd}, etc., orders got by including x^2, $x^3 \ldots\ldots$ When one or more of the quantities b_1, c_1, etc., vanish, cases of contact of higher orders arises, and this leads to the discussion of points of inflexion and cusps.

I may pass over the discussion of asymptotes which is contained in the next section of the Calculus, with the remark that points at infinity are given by roots of the equation $p + q + r = 0$, and the asymptotes or tangents at these points are discussed by the method which has already been illustrated in the case of finite tangents.

Curves and Surfaces in Space.

A curve in space is represented by $pA + qB + rC + sD$ where p, q, r, s are functions of a single variable.

If p is a function (of v say) of the n^{th} degree, the n roots of $p = 0$, substituted in $qB + rC + sD$ give the n points where the curve meets the plane BCD.

The line joining A to the point $pA + qB + rC + sD$ meets the plane BCD in the point $qB + rC + sD$. Hence the curve may be formed as the intersection of any two of the four cones such as that which has A as vertex and $qB + rC + sD$ as generating curve.

The expression for the tangent and the osculating plane at any point are obtained by a method practically identical with that used in the case of plane curves.

A surface is represented by $pA + qB + rC + sD$ where p, q, r, s, are functions of two independent variables.

If p, q, r, s, are functions (say of v and w) of the n^{th} degree, the surface will in general be of a higher order than the n^{th}. For if we determine v and w by means of the equations $p = 0$, $q = 0$, and substitute the values obtained in $rC + sD$, we shall obtain the points in which CD meets the surface. Now $p = 0$, $q = 0$, will have in general more than n solutions.

The tangent-plane at a point of a surface is discussed as follows:—

Let $pA + qB + rC + sD$ be the surface, where p, q, r, s, are functions of v and w. If v' and w' be particular values and x and y infinitely small increments, p becomes

$$p' + \frac{dp'}{dv'}x + \frac{dp'}{dw'}y. \text{ Hence } \Sigma\left\{(p' + \frac{dp'}{dv'}x + \frac{dp'}{dw'}y)A\right\}$$

is an infinitely near point. If now x and y are taken to be any two independent variables, this will represent the tangent plane at $(p,' q', r', s'.)$

The conditions that ABC be a tangent plane are

$$s'=0,\ \frac{ds'}{dv'}=0,\ \frac{ds'}{dw'}=0.$$

Since s vanishes when $v=v'$, $w=w'$, assume $s=b(v-v')+c(w-w')$ where b and c are functions of v and w.

$$\begin{aligned} \therefore\ \frac{ds}{dv} &= b+(v-v')\frac{db}{dv} \qquad +(w-w')\frac{dc}{dv} \\ \frac{ds}{dw} &= \qquad (v-v')\frac{db}{dw}+c+(w-w')\frac{dc}{dw} \end{aligned}$$

Hence b and c vanish when $v=v'$, $w=w'$.
Hence s must be of the form $f(v-v')^2+g(v-v')(w-w')+h(w-w')^2$.

If A is the point of contact, the equation to the surface must be of the form

$$\begin{aligned} a\mathrm{A}+[b(v-v')+c(w-w')]\mathrm{B}+[d(v-v')+e(w-w')]\mathrm{C} \\ +[f(v-v')^2+g(v-v')(w-w')+h(w-w')^2]\mathrm{D}. \end{aligned}$$

where $a, b, \ldots h$ are functions of v and w.

In considering the nature of the contact, we may assume $a, b, \ldots h$ to be constants and $v-v'$ and $w-w'$ to be infinitely small. Putting $t=[b(v-v')+c(w-w')]/a$; $u=[d(v-v')+e(w-w')]/a$ the expression becomes $\mathrm{A}+t\mathrm{B}+u\mathrm{C}+(it^2+ktu+lu^2)\mathrm{D}$.

Put $\mathrm{A}+t\mathrm{B}+u\mathrm{C}\equiv\mathrm{Q}$ and $\mathrm{Q}+(it^2+ktu+lu^2)\mathrm{D}\equiv\mathrm{P}$; then Q is a point in the tangent-plane ABC infinitely near the point of contact A; while P is a point on the surface, lying in the line QD and distant from Q by a quantity of the second order. Further P lies towards D or away from D according as $it^2+ktu+lu^2$ is positive or negative. Hence if $k^2<4il$, the surface is altogether on one side of the tangent plane in the neighbourhood of A ; if $k^2>4il$ the surface crosses the tangent-plane. This leads to the distinction between synclastic and anticlastic surfaces. There is also a limiting case in which $k^2=4il$, when the surface touches the plane along a curve.

Möbius shows that, if the restriction that t and u be infinitely small be removed, then $\mathrm{A}+t\mathrm{B}+u\mathrm{C}^2+(it^2+ktu+lu^2)\mathrm{D}$ represents a quadric surface. To examine the different forms of quadric surface, put $s=lu^2+ktu+it^2+u+t+1$.

We may show that $s=\frac{1}{4l}\left[v^2+\frac{w^2+e}{4(4il-k^2)}\right]$

where v and w are linear functions of t and u and where $e=16l(4il-k^2-i+k+l)$.

The points at infinity are obtained by making s zero; hence there are no such points if $4il>k^2$ and e is positive. In this case the surface is an ellipsoid.

If $4il>k^2$, but e is negative, the surface does not cross the tangent plane at the point of contact, and there are real points at infinity (Hyperboloid of two sheets).

If $4il<k^2$, we get a hyperboloid of one sheet.

If $e=o$, we get an elliptic paraboloid if $4il>k^2$, a hyperbolic paraboloid if $4il<k^2$.

Möbius next proceeds to consider some simpler forms of the expressions for quadric surfaces and to deduce some of the properties of such surfaces; and then devotes a few pages to developable surfaces.

The next section of the Calculus is devoted to the problem of determining the equation to a curve when its barycentric expression is given; and to the converse problem.

The first of these problems is easily soluble; but, as has been already pointed out, the second is only soluble in particular cases. I shall only point out how the first of these problems is solved in the case of plane curves.

Let A, B, C, be the vertices of the triangle of reference, and let the co-ordinates of these points be (a, a'), (b, b'), (c, c') with references to a pair of Cartesian Axes.

Let $P\equiv pA+qB+rC$ and let the co-ordinates of P be (x, y). If P′, A′, B′, C′, are the feet of the ordinates from P, A, B, C, then, by the fundamental theorem of the Calculus,

$$p.AA'+q.BB'+r.CC'=(p+q+r).PP'$$

$$\left.\begin{array}{ll}\textit{i.e.,} & pa'+qb'+rc'=(p+q+r)y \\ \text{and similarly} & pa+qb+rc=(p+q+r)x\end{array}\right\}$$

Thus we get x and y in terms of p, q and r. If p, q and r are given functions of a variable v, so that $pA+qB+rC$ represents a curve, we get the equation to the curve by eliminating v between the equations that give x and y.

This section concludes the first part of the Barycentric Calculus; the second part is devoted to the discussion of certain relations between figures and of some geometrical theorems connected with these relations. This part also contains a lengthy discussion of the theory of anharmonic ratio and what are called geometrical nets.

Möbius considers five different kinds of relations between figures. The first of these he describes by the phrase equality and similarity (Gleichheit and Aehnlichkeit); but as the second relation is similarity and the fourth is equality, it will be better to employ the single term congruence. The five relations then are (1) Congruence, (2) Similarity, (3) Affinity, (4) Equality, (5) Collineation. The most general of these relations is the last, all the others being particular cases of it; and towards the end of his discussion, Möbius shows that of any two plane figures between which the relation of collineation subsists, one may be exhibited as a conical projection of the other. The relation of collineation is thus the same as that of homology or homography, of which a discussion was given before the date of the Barycentric Calculus by Poncelet, and after that date by Chasles; and an account of the theory of homology is now contained in many geometrical text-books. But as Möbius's treatment is based on the methods of his Calculus, and for the sake of completeness, it may be well to give some account of his theory of these five geometrical relations.

The account of congruence and similarity I may pass over with the remark that he enumerates the number of conditions required to determine certain figures. The results of these enumerates I shall bring together in a table at the end of this account of his discussion.

The relation of affinity is defined as follows:—

Let ABC and A′B′C′ be any two triangles, and let D and D′ be two points determined with reference to these triangles by means of the relations $D \equiv aA + bB + cC$ and $D' \equiv aA' + bB' + cC'$, then D and D′ are corresponding points. From this it is obvious that if one system of points be given then the corresponding system of points will be determined when three of the points are given; but these three points may be given arbitrarily.

If P, Q, R, S, are any four points of the one system, P′, Q′, R′, S′, the corresponding points in the other system, we shall have relations of the form

$$\left.\begin{aligned} pP + qQ + rR + sS &= 0 \\ pP' + qQ' + rR' + sS' &= 0 \end{aligned}\right\}$$

and $\therefore\ p : q : r : s = \triangle QRS : \triangle PSR : \triangle SPQ : \triangle RQP$
$= \triangle Q'R'S' : \triangle P'S'R' : \triangle S'P'Q' : \triangle R'Q'P'$.
Hence the ratio PQR : P′Q′R′ is constant.

It is next shown that the above definition is equivalent to that given in terms of Cartesian co-ordinates by Euler in his *Introductio in Analysin Infinitorum;* and then the theory is extended to space of three dimensions by means of the definition that E and E′ are corresponding points if $E \equiv aA + bB + cC + dD$ and $E' \equiv aA' + bB' + cC' + dD'$.

It is easily shown, by a method perfectly analogous to that used above for plane figures, that corresponding solids are to one another in a constant ratio.

Further if in one figure, three points are collinear or four points coplanar, then the same will be true of the corresponding points in the other figure; and if A, B, C, are collinear points and A′, B′, C′, the corresponding points, then $AB : BC = A'B' : B'C'$.

It has been pointed out that in two plane figures that bear to one another the relation of affinity, if P, Q R, and P′, Q′, R′, are corresponding points, then $\triangle PQR : \triangle P'Q'R' = \text{const}$ (m, say), and in space of three dimensions if P, Q, R, S, and P′, Q′, R′, S′, are corresponding points, then $\text{vol.}PQRS : \text{vol.}P'Q'R'S' = \text{const}$ (n, say). If m is unity, we get the relation of equality for plane figures; and if n is unity, for solid figures. To obtain two such figures, it is only necessary to make the triangles of reference equal in area in the one case, and the tetrahedrons of reference equal in the other case.

Möbius works out a number of cases of the calculation of parts of figures in plane and solid geometry, in connection with these relations, and applies this theory to prove some properties of conic sections and of quadric surfaces. I select the following as an example.

Given five points ABCDE (fig. 4) in a plane, and given the four ratios

$$\left.\begin{aligned} ABC : CDA &= p \\ BCD : DAB &= q \\ ABE : CDE &= r \\ DAE : BCE &= s \end{aligned}\right\} \quad \begin{array}{c}\text{to calculate the ratio} \\ DBE \div ACE = t.\end{array}$$

Assume $d''D = dA + d'B + C,\ e''E = eA + e'B + C$ ... (1), (2), so that $d'' = 1 + d + d'$ and $e'' = 1 + e + e'$.

Eliminating first C and then A, we get

$$(d-e)\mathrm{A}+(d'-e')\mathrm{B}-d''\mathrm{D}+e''\mathrm{E}=0, \quad \ldots \quad \ldots \quad (3)$$
$$(de'-ed')\mathrm{B}+(d-e)\mathrm{C}+ed''\mathrm{D}-de''\mathrm{E}=0. \quad \ldots \quad \ldots \quad (4)$$

From (1) we get the relations

$$\mathrm{BCD} : \mathrm{DCA} : \mathrm{DAB} : \mathrm{ABC} = d : d' : 1 : d''$$

with similar results from (2), (3), and (4).

These enable us to express p, q, r, s and t in terms of d, d', e, e'; viz., we get

$$p=-d''/d',\ q=d,\ r=d''/(de'-d'e),\ s=(e'-d')/ed'' \text{ and } t=(e-d)/e'd''.$$

On eliminating d, d', e, e' from these four equations, we get

$$t=(1+p)^2(1-q^2rs)/(1+q)^2(p^2s-r).$$

The next chapter is devoted to the discussion of anharmonic ratio (Doppelschnittsverhältniss) (ratio bissectionalis). It is here that this theory was first fully discussed; but as the subject is now well known, I may pass it over altogether.

The next subject considered is that of geometrical nets in a plane and in space of three dimensions. If we start with four points in a plane, and join them in every possible way, we get three new points. If we draw the three straight lines joining these three points in pairs, we obtain three new points as the intersections of these lines with lines already drawn in the figure. Proceeding in this way we may get an infinite number of points and straight lines in the plane. The figure formed by all these straight lines is called a plane geometrical net; and the analogous figure in space of three dimensions may be formed by starting with five points in space.

It is easily shown by mathematical induction that if A, B, C, D are the four principal points of a plane net, and if $\mathrm{D}\equiv a\mathrm{A}+b\mathrm{B}+c\mathrm{C}$, then every other point of the net may be expressed in the form $\lambda a\mathrm{A}+\mu b\mathrm{B}+\nu c\mathrm{C}$ where λ, μ, ν are rational numbers, and are independent of a, b, and c. Further, by calculating the anharmonic ratio of four collinear points, we may show that its value is independent of a, b, c; and therefore this anharmonic ratio does not depend on the position of A, B, C, D, but only on the steps by which the four points considered have been derived from A, B, C, D. The converse to the theorem quoted above is then proved, namely, that any point $\mathrm{P}\equiv\lambda a\mathrm{A}+\mu b\mathrm{B}+\gamma c\mathrm{C}$ where λ, μ, ν are rational numbers is a point in the net. In proving this we may assume λ, μ, ν, to be integers. Constructions are given for the points $2aA+bB+cC$, $3aA+bB+cC$, etc., in succession, and this shows that we may construct the point $\lambda aA+bB+cC$. Then constructions for the points $\lambda aA+2bB+cC$, $\lambda aA+3bB+cC$ are given, showing that we may construct $\lambda aA+\mu bB+cC$, and then similarly that we may construct $\lambda aA+\mu bB+\nu cC$. In order to apply this theory to the last of the geometrical relations, that of collineation, Möbius proves (what is really obvious) that every point of the plane is either a point of the net, or infinitely near to a point of the net. He proves a number of theorems in plane and solid geometry by means of the theory of nets, and then proceeds to consider what he calls a "Vieleckschnittsverhältniss." This word may be translated by the phrase "polygonal cross-ratio." There is a triangular cross-ratio, tetragonal cross-ratio, and so on. If P, Q, R, are points in the sides BC, CA, AB of a triangle, then

$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB} \text{ is a triangular cross-ratio.}$$

It is shown that, just as in the case of an ordinary cross-ratio, if A, B, C, P, Q, R, are points of a net, then the value of the triangular cross-ratio is independent of the positions of the four principal points of the net.

The defining property of the last kind of geometrical relation considered, that of collineation, is this, that to three collinear points in the one figure shall correspond three collinear points in the other figure. Two geometrical nets (both plane or both solid) satisfy this definition, and are the most general figures that can satisfy it. For if we start with two tetrads of corresponding points A, B, C, D, and A′, B′, C′, D′, we may obtain from these by the first step, three new points in each figure, E, F, G, and E′, F′, G′, and by the definition of collineation E′ must correspond to E, F′ to F, and G′ to G; and so on for all the other points of the two nets. Now we may put $D\equiv aA+bB+cC$ and $D'\equiv a'A'+b'B'+c'C'$, where a, b, c, a', b', c', are arbitrary, and any point of the first net may be written $P\equiv\lambda aA+\mu bB+\nu cC$, and the corresponding point in the second net will be $\lambda a'A'+\mu b'B'+\nu c'C'$. Hence this is the analytical definition of the relation of collineation. If we suppose λ, μ, ν to be functions of a single variable, we shall get two curves which are subject to this relation; and it is obvious that any two corresponding curves are of the same order. From what has been proved above it follows that a cross-ratio or a polygonal cross-ratio in one figure is equal to the cross-ratio or the polygonal cross-ratio formed by the corresponding points in the other figure.

As has been mentioned above, Möbius shows that of any two figures between which this relation of collineation subsists either may be regarded as the conical projection of the other. Hence the relation of collineation is identical with that of homography, or if the relative position of the two figures be restricted in a certain way, that of homology. The above definition is easily extended to space of three dimensions; and, among his illustrations of the theory, Möbius shows that by the method of collineation, any one conic section may be transformed into any other, but that a similar theorem does not hold good of quadric surfaces. These divide themselves into two groups, the first of which includes the ellipsoid, the hyperboloid of two sheets and the elliptic paraboloid, and the second the hyperboloid of one sheet and the hyperbolic paraboloid; in other words, the first class includes the non-ruled surfaces, and the second class the ruled surfaces. Any one surface of one of these classes may be transformed into any other of that class; but one from one class cannot be transformed into one from the other class.

Möbius next explains what he calls the contracted (abgekürzte) Barycentric Calculus. This contracted form is applicable to problems which are true of all figures collinearly related. The contraction consists in omitting the a, b, c, from the equations, and taking for the four principal points A, B, C and $D \equiv A + B + C$. This contracted method is applied to certain problems connected with the triangle and the tetrahedron.

Möbius next proceeds to classify certain geometrical problems according to the geometrical relation (congruence, similarity, etc.) under which they come. If there are n points in a straight, in a plane or in space, we may want to calculate the length of a line, the ratio of two lines, an area, etc. The following table gives the number of conditions necessary in each case :—

	I.	II.	III.	
Congruence	$n-1$	$2n-3$	$3n-6$	
Similarity .	$n-2$	$2n-4$	$3n-7$	Ratios.
Equality .	$n-1$	$2n-5$	$3n-11$	Portions of space (areas, etc.)
Affinity .	$n-2$	$2n-6$	$3n-12$	Ratios between portions of space.
Collineation	$n-3$	$2n-8$	$3n-15$	Polygonal cross-ratios.

The meaning of this table may be illustrated by taking (say) the second column under the head of "affinity." If there are n points in a plane, and if $2n-6$ independent ratios of areas of triangles or polygons formed by certain of these points are given, then all the figures that can be constructed to satisfy these conditions will have the relation of affinity to one another; and from the $2n-6$ given ratios any other ratio may be calculated. Many illustrations are given by Möbius, and one of these has been quoted above.

Möbius points out that problems connected with congruence and similarity, *e.g.*, such as involve angles, cannot be solved by means of the Barycentric Calculus at all; those that are true of all figures which satisfy the relation of affinity can be solved by the ordinary form of the Calculus; while those that are true of all figures collinearly related may be solved by the contracted form. This forms a three-fold classification of geometrical theorems.

The third and last part of the work is concerned with the application of the Calculus to the investigation of certain properties of the Conic Sections and with a discussion of the principle of duality. From this part I shall only select a few examples by way of illustration. The first question considered is the determination of a conic section by means of five given points or five given tangents.

As we have seen already, the general expression for a conic circumscribing the triangle of reference is

$$a(v-\beta)(v-\gamma)\mathrm{A}+b(v-\gamma)(v-\alpha)\mathrm{B}+c(v-\alpha)(v-\beta)\mathrm{C}.$$

This expression may be reduced to the form

$$fx\mathrm{A}-gx(1-x)\mathrm{B}+h(1-x)\mathrm{C} \quad \ldots \quad \ldots \quad (1)$$

where

$$x = -\frac{\alpha-\beta}{\gamma-\alpha}\cdot\frac{v-\gamma}{v-\beta};\; f = a(\beta-\gamma),\; g = b(\gamma-\alpha),\; h = c(\alpha-\beta).$$

Let $D \equiv aA + bB + cC$, then the conditions that the conic pass through this point are

$$\frac{f}{1-x} : -g : \frac{h}{x} = a : b : c,$$

whence $$f/a + g/b + h/c = 0.$$

If $E \equiv a'A + b'B + c'C$ is the fifth point, we must also have $f/a' + g/b' + h/c' = 0$; and from these two equations we get

$$f : g : h = \left(\frac{1}{bc'} - \frac{1}{b'c}\right) : \left(\frac{1}{ca'} - \frac{1}{c'a}\right) : \left(\frac{1}{ab'} - \frac{1}{a'b}\right)$$

$$= a(\beta-\gamma) \quad : \quad b(\gamma-\alpha) \quad : \quad c(\alpha-\beta),$$

where $$a' = a/\alpha,\; b' = b/\beta,\; c' = c/\gamma.$$

Hence the expression for the conic through the five points A, B, C, D, E, is

$$a(\beta-\gamma)xA - b(\gamma-\alpha)x(1-x)B + c(\alpha-\beta)(1-x)C;$$

and the values of x that give these points are

$$1,\; \beta,\; 0,\; -(\alpha-\beta)/(\gamma-\alpha) \text{ and } -\gamma(\alpha-\beta)/\beta(\gamma-\alpha).$$

Putting $(\alpha-\beta)/(\gamma-\alpha) = m$, and $\therefore\ (\beta-\gamma)/(\gamma-\alpha) = -1+m$, we get the expression $a(1-m)xA + bx(1-x)B + cm(1-x)C$, which, if we suppose m to be arbitrary, will represent any conic passing through A, B, C, D.

Taking the form (1) we may determine the condition that it represent an ellipse, a hyperbola or a parabola, by considering the sum of the co-efficients. This sum is

$$fx - gx(1-x) + h(1-x) = gx^2 - (g+h-f)x + h.$$

The discriminant of this is

$$f^2 + g^2 + h^2 - 2(gh + hf + fg).$$

The condition for a parabola is

$$g + h - f = 2\sqrt{gh} \text{ or } f = (\sqrt{g} - \sqrt{h})^2.$$

Putting $e = \sqrt{(h/g)}$, the expression for a parabola circumscribing ABC becomes

$$(1-e)^2xA - x(1-x)B + e^2(1-x)C.$$

Now take a fourth point $D \equiv aA + bB + cC$, and put $a + b + c + d = 0$, then the equation to determine e, so that the parabola shall pass through D reduces to

$$(1 - e)^2/a + 1/b + e^2/c = 0$$

that is,
$$(a + c)be^2 - 2bce + (a + b)c = 0.$$

The discriminant of this reduces to the form $abcd$.

Now if this expression is positive, two of the factors must be positive and the other two negative, and therefore each of the points ABCD will be outside the triangle formed by the other three. On the other hand, if the expression is negative, three of the factors must have the same sign, and therefore one of the points will be inside the triangle formed by the other three. In the first case it will be possible to construct two parabolas to pass through ABC and D, and in the other case no such parabola can be described.

Using the form $a(1 - m)xA + bx(1 - x)B + cm(1 - x)C$ ... (2), we find that the discriminant of the sum of the co-efficients is

$$M \equiv (b + c)^2 m - (c + a)^2 m(1 - m) + (a + b)^2(1 - m).$$

The discriminant of M, considered as a function of m, is $abcd$; and hence the expression (2) may represent a parabola if $abcd$ is positive, as we have already shown. If, on the other hand, $abcd$ is negative M will always have the same sign, and (2) can represent only hyperbolas or only ellipses. When $m = 0$, M becomes $(a + b)^2$, and hence if $abcd$ is negative (2) can represent only hyperbolas. This case, it is to be remembered, occurs when one of the four points A, B, C, D, lies within the triangle formed by the other three.

Of five points in a plane there will always be four such that each of them lies outside the triangle formed by the other three. Let these be A, B, C, D (fig. 5).

If E lies in one of the spaces H the curve must be a hyperbola; if in one of the spaces G it may be either a hyperbola or an ellipse. Construct the two parabolas through ABCD (fig. 6). These parabolas must lie altogether in the spaces G. If we letter the spaces as in the figure, we see that, if E lies in one of the spaces L, the curve is an ellipse; if E lies in one of the other spaces, the curve is a hyperbola. This follows from the continuity of M and from the fact that M is zero when E is on one of the parabolas.

As a last illustration of the application of the Calculus we may take the following :—

In the triangle ABC (fig. 7) is inscribed the triangle A′B′C′, in this the triangle A″B″C″, and so on, so that the three sets of corresponding vertices lie on straight lines which pass through a fixed point D. In the triangle ABC, a conic is inscribed touching the sides at A′, B′, C′, in A′B′C′ a conic is inscribed touching at A″, B″, C″. To show that the centres of all these conics lie in a straight line passing through D.

Let $$dD = iA + kB + lC,$$

and put $$k + l = i',\ l + i = k',\ i + k = l'$$
$$k' + l' = i'',\ l' + i' = k'',\ i' + k' = l'', \text{ etc.}$$

Then $$i'A' = kB + lC,\ k'B' = lC + iA,\ l'C' = iA + kB.$$

Similarly $$i''A'' = k'B' + l'C', \text{ etc., etc.}$$

Now the centre of the conic which is inscribed in ABC and touches the sides at the points $kB + lC$, $lC + iA$, $iA + kB$, is

$$\begin{aligned} mM &= i(k+l)A + k(l+i)B + l(i+k)C \\ &= ii'A + kk'B + ll'C, \end{aligned}$$

where $$m = ii' + kk' + ll'.$$

Similarly the centre of the second conic is

$$\begin{aligned} m'M' &= i'(k' + l')A' + k'(l' + i)B' + l'(i' + k')C \\ &= i'i''A' + k'k''B' + l'l''C' \\ &= i''(kB + lC) + k''(lC + iA) + l''(iA + kB) \\ &= i(k'' + l'')A + k(l'' + i'')B + l(i'' + k'')C \\ &= ii'''A + kk'''B + ll'''C, \end{aligned}$$

where $$m' = i'i'' + k'k'' + l'l'' = ii''' + kk''' + ll'''.$$

From these we get

$$m'M' - mM = i(i''' - i')A + k(k''' - k')B + l(l''' - l')C.$$

Now

$$i''' - i' = k'' + l'' - i' = l' + i' + i' + k' - i' = i' + k' + l' = 2d;$$

and hence

$$m'M' - mM = 2d(iA + kB + lC) = 2d^2D.$$

Hence M, M′ and D are collinear and hence all the centres lie on a straight line passing through D.

Möbius calculates the values of the successive ranges formed by these centres, and finds that

$$(\mathrm{DMM'M''}) = 5,\ (\mathrm{MM'M''M'''}) = (\mathrm{M'M''M'''M''''}) = \text{etc.} = \frac{25}{21}.$$

The last two chapters of the Calculus are devoted to a discussion of the principle of duality. This principle, which is now so well known, was only being investigated when Möbius wrote in 1827. As a consequence of the theory of reciprocal polars, it was first stated by Poncelet in the year 1818, and as an independent principle by Gergonne in 1826, and the Barycentric Calculus was published in 1827. In his preface Möbius states that before the part of his work dealing with duality was sent to press he had heard of the papers of Poncelet and Gergonne, but had not seen them. It is unnecessary to give an account of his treatment of the subject; but it may be well to state that he enunciates the principle quite generally as a characteristic property of space and treats the subject independently of the properties of conic sections.

Second Meeting, December 11, 1891.

Professor J. E. A. STEGGALL, M.A., President, in the Chair.

Note on an approximate fractional expression for the expansion of $(1+x)^n$ to any odd number of terms.

By Professor J. E. A. STEGGALL.

In Bonnycastle's *Arithmetic* a rule is given for finding any root of a number by approximation which in substance reduces to this statement:—If a and b are nearly equal, then the n^{th} root of a/b is nearly equal to

$$\frac{(n+1)a+(n-1)b}{(n+1)b+(n-1)a};$$

or, if we call a/b, $1+x$, we have $(1+x)^{\frac{1}{n}} = \dfrac{2n+(n+1)x}{2n+(n-1)x}$ ... (1)

For example, let $a=2$, $b=1$, $n=3$, then an approximation to the cube root of 2 is

$$\frac{4.2+2.1}{4.1+2.2}=\frac{5}{4},$$

a first approximation which can be used as a basis for a second thus:—

$$2^{\frac{1}{3}} = \frac{5}{4}\left(\frac{128}{125}\right)^{\frac{1}{3}},$$

and applying the rule

$$2^{\frac{1}{3}} = \frac{5}{4} \cdot \frac{4.128 + 2.125}{4.125 + 2.128}, \text{ nearly,}$$

$$= \frac{635}{504} \text{ nearly.}$$

This result is obtained with little labour, and is correct to six decimal places.

It is clear that we may test the accuracy of the approximation by expansion, and we accordingly find that the difference in the two sides of (1) is

$$\frac{x^3(n^2 - 1)}{12n^3} \dots$$

an error in the given example of about

$$\frac{3^3.8.5}{125^3.\ 12.\ 3^3.\ 4} \text{ or } \cdot 0000001.$$

A general problem is at once presented, namely, to find a fraction

$$\frac{1 + a_1x + a_2x^2 + \ \dots\ + a_rx^r}{1 + b_1x + b_2x^2 + \ \dots\ + b_rx^r} \qquad \dots \qquad \dots \qquad (2),$$

which on expansion shall give the value of $(1 + x)^n$ correct to $(2r + 1)$ terms. Of course for a direct solution we have $2r$ linear equations to determine the *a's* and *b's*, and these equations I succeeded in solving. At the time, I had no opportunity of consulting Gauss, but since the paper was read I find that my result is included in one of his on hypergeometrical series, and is an exact theorem: that is to say, the numerator and denominator of the fraction (2) are the first $r + 1$ terms of two hypergeometrical series of which the next r terms are zero, and the finite terms then recommence and continue *ad infinitum*. This of course explains the accordance to $2r + 1$ terms when the division of and by the first $r + 1$ terms is effected.

The final result is

$$(1 + x)^n = \frac{1 + \frac{(n + r)r}{2r}x + \frac{(n + r)(n + r - 1)r(r - 1)}{2r(2r - 1)1.2}x^2 + \dots \textit{ad infinitum}}{1 - \frac{(n - r)r}{2r}x + \frac{(n - r)(n - r + 1)r(r - 1)}{2r(2r - 1)1.2}x^2 + \dots \textit{ad infinitum}}$$

where the factor $(r - r)$ in the $(r + 2)^{th}$ terms causes evanescence until the introduction of $(2r - 2r)$ in the denominator of the $(2r + 2)^{th}$ terms produces a finite, thereafter, result.

It may be interesting to write down a few cases for the example already considered, namely $(1+x)^{1/3}$;

the first result is $$\frac{3+2x}{3+x}$$

the second $$\frac{1+7x/6+7x^2/27}{1+5x/6+5x^2/54}$$

the third $$\frac{1+5x/3+7x^2/9+7x^3/81}{1+4x/3+2x^2/9+2x^3/81}.$$

In Gauss's notation the identity is

$$(1+x)^n = \frac{F(-n-r,\ -r,\ -2r,\ -x)}{F(\ n-r,\ -r,\ -2r,\ -x)},$$

of which a more general case will be found in his *Methodus nova integralium valores per approximationem inveniendi*, §§ 40-42.

On the history and degree of certain geometrical approximations.

PART I.

By A. J. PRESSLAND, M.A.

According to Luther's translation 1 Kings vi. 31 should read—"At the entrance of the choir he made two doors with pentagonal door-posts." This is probably a wrong translation, for nowhere on Asiatic monuments of this time has a pentagon been found. Prof. A. Merx in Cantor's *Vorlesungen über Geschichte der Mathematik*, Vol. i., p. 91.

It is probable that the construction of the regular pentagon is due to the Pythagorean school as a consequence of I. 47. (*Ibid*, p. 151.)

Various approximations to the arc of a regular pentagon in a circle have been given. Thus (fig. 8) a diameter AB of a circle is divided into five equal parts at E, D, etc. On AB an equilateral triangle is erected giving the vertex C. CD is drawn to cut the circumference in P, and AP is assumed to be the arc of the regular inscribed pentagon. By dividing AB into n equal parts and joining C to the second of these from A, an approximation to the arc of the regular n-gon is obtained.

The first mention of this construction is in *Les fortifications* of Chevalier Antoine de Ville (achevé d'imprimer, 1er Août, 1628). In this, however, the point C is joined to the point E and the resulting arc doubled. On page 29 the following occurs: "Ce probleme ne

se demonstre point : et au calcul il ne revient pas tout à fait precisément juste principalement aux figures qui ont grand nombre de costez . . . il est à estimer pour la facilité et justesse plus grande que de tous les autres qui ont esté escrits pour ce sujet."

Bion in his *Instruments*, pp. 19, 20 (1703) gives this method, and ascribes it to De Ville. He says that by De Ville's method the angle at the centre is 44′ too large for the pentagon and 5′ too large for the heptagon, from which point the difference gradually increases. He follows a variation given by Bosse in his *Traité des pratiques géométrales et perspectives*, p. 62 (1665) (privilège 16th May 1653), in which C is joined to D. This improvement, he says, makes the angle at the centre 2′ too small for the pentagon and 6′ too large for the heptagon.

The following note occurs in Christian Wolf's *Elementa Matheseos Universae* (1730) :—

Pentagoni Decagoni et Quindecagoni constructionem tradunt Euclides IV. 11, 16 : XIII., 10 et Ptolomaeus Almag. lib. 1 *c* 9 *fm* 8 : equidem et heptagoni enneagoni et hendecagoni constructiones Geometricae passim apud autores practices imprimis occurrunt : sed a rigore demonstrationis abhorrent. Joh. Carolus Renaldinus lib. 2 de Resolut. et Composit. Mathem. *f* 367 omnium polygonorum describendorum regulam catholicam praescribit, passim geometriis practicis insertam [De Ville's method].

It is possible that Wolf was misled by Sturm, for on page 233 of Sturm's *Mathesis Juvenilis* (1702), De Ville's method is ascribed to Renaldinus, as also on p. 38 of his *Mathesis Enucleata.*

This solution appears in John Ward's *Young Mathematician's Guide*, 1706. T. S. Davies, in his edition of Hutton's Course of Mathematics, ascribes it to Thomas Malton, in whose *Royal Road to Geometry* it occurs, Part II., p. 40 (1793).

In Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*, 6^{me} *Édit.*, p. 279, a short history is given, and the following formula for the length of AP in a circle of radius unity

$$x^2 = 2 - \frac{n-4}{2} \cdot \frac{3n + \sqrt{n^2 + 16n - 32}}{(n-1)^2 + 3} .$$

In the *Nouvelles Annales de Mathématiques*, tome XII., p. 77 (1853), M. Housel gives the following table :—

Number of Sides.	Approximate Angle at Centre.	Correct Angle at Centre.
3	120°	120°
4	90°	90°
5	71° 57′ 12″	72°
6	60°	60°
7	51° 31′ 5″	51° 25′ 43″
8	45° 11′ 14″	45°
9	40° 16′ 40″	40°
10	36° 21′ 24″	36°
11	33° 8′ 52″	32° 43′ 38″
12	30° 29′ 45″	30°
13	28° 12′ 30″	27° 41′ 32″
14	26° 15′ 48″	25° 42′ 52″
15	24° 34′ 30″	24°
16	23° 5′ 54″	22° 30′
17	21° 47′ 12″	21° 10′ 35″

A modification of the construction is to be found in Hawney's *Mensuration*, and in Adams' *Mathematical Instruments* (1803). In this the perpendicular is made 1·75 times the radius. The centre angle so obtained is 72° 1′ 7·9″.*

In Hutton's *Miscellanea Mathematica*, pp. 311-12 (1775) a discussion of this problem is given by Geometricus, in which the problem is presented in the following form :—

Is it possible to find a point in OC so that CDP being drawn the line DO shall bear a constant ratio to the arc PQ?

§ 2. The construction given in fig. 9 is due to Albrecht Dürer being given in his *Underweysung der Messung mit dem Zirckel*, fig. 16 (1525). He also gives the construction from the Almagest. This construction is effected with a single opening of the compasses, a condition observed by various writers and afterwards amplified by Tartaglia. The four points H, A, B, F are concyclic. If a be the centre of this circle the side AB subtends 71° 38′ 6·3″ at a.

With regard to this construction, Daniel Schwenter, in his *Deliciae Mathematicae*, Part III., page 181 (1636), says: "Dieses

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[*] The very close approximations are marked with an asterisk.

Fünfeck berechnet Benedict in Epist. f. 349 und erweiset dass es ganz richtig ist."

But in the new edition of his *Geometria practica* (Nürnberg 1677) edited by George Andreas, the following note occurs: "Es ist zu mercken dass dasselbige nur gleiche Seiten hat aber nicht gleiche Winckel, wie solches vom M. J. P. Seligen und nach ihme von dem Clavio erwiesen worden."

In a list of authors given in the *Deliciae Mathematicae* reference is made to M. Johannes Praetorius, the only author having the above initials.

In Clavius's *Opera Mathematica*, Vol. II., pp. 310–11 (1611) the angles A and B are calculated to be 108° 22′, F and H 107° 2′ and the remaining angle 109° 12′.

Chasles in his *Aperçu historique*, p. 530, says:

Durer apprend à construire un pentagone régulier sur un côté donné, et sa construction a cela de remarquable qu'elle se fait avec une seule ouverture de compas; mais elle n'est qu'approximative, et la figure qui a conservé le nom de pentagone de Durer n'a pas tous ses angles égaux ainsi que l'ont démontré J. B. de Benedictis* et Clavius† dans le siècle suivant.

§ 3. In Schwenter's *Deliciae Mathematicae* the following construction occurs (fig. 10).

Take a line CD and on it describe an equilateral triangle CBD. Let CE, ED be two sides of a square having CD for diagonal. Bisect EB at F and assume F as the centre of the regular pentagon on CD as base.

He further remarks:—By taking off along AB produced parts equal to BF, the centres of regular polygons, on CD as side, of from 7 to 12 sides may be obtained.

The angle subtended by CD at F is 72° 24′ 43″·4.

§ 4. A method ascribed by Schwenter to Augustine Hirschvogel (fig. 11) consists in describing three sides of a regular hexagon *adec*. With *c* as centre and *ca* as radius describe an arc cutting *ed* produced in *g*. Assume *bg* the side of the pentagon inscribed in *adec*.

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* *Diversarum speculationum mathematicarum et physicarum Liber* (1585)

† *Geometria practica*, lib. VIII. prop. 29.

The angle subtented at the centre by a chord equal to *bg* is 72° 22′ 32″·4.

As Daniel Schwenter was professor of Oriental languages as well as of Mathematics, it is possible that some of these solutions are taken from Oriental sources.

§ 5. Cantor says that from Arabic remains it seems that Archimedes wrote a book on the inscription of the heptagon in a circle which is lost.

The method of inscribing a regular heptagon in a circle by taking half the side of the inscribed equilateral triangle for the side of the heptagon is ascribed by Cantor, p. 640, to Abû'l Wafâ, a Persian of Khorassan (940-998). See Wöpcke *Journal Asiatique*, Feb. and March 1855, pp. 329, 332. This method gives the value 51° 19′ 4″ for the angle subtended at the centre instead of 51° 25′ 43″.

De Ville's method makes the centre angle 51° 31′ 5′.

A method ascribed by Clavius to Carolus Marianus Cremonensis is to produce a radius CB of a circle one-fourth of its length to A, (fig. 12). With A as centre and radius equal to the radius of the given circle, describe an arc cutting the given circle in D, and assume BD the side of the regular inscribed heptagon.

The length BD can be proved equal to half the side of the inscribed equilateral triangle (Pappus III. 5) and depends on the close approximation of $\cos 2\pi/7$ to 5/8.

If the method of § 3 be followed, the angle subtended by CD at the assumed heptagon centre would be 50° 58′ 3″.

It is to be noted that the regular heptagon can be inscribed in a circle by means of Peaucellier-cells. See *Educational Times Reprint*, Vol. LV., Question 10865, p. 61, by Mr A. A. Robb.

§ 6. The following with slight alterations is given by Albrecht Dürer in the folio of 1525 (fig. 13).

Describe a circle and draw three radii AB, AC, AD inclined to each other at 120°. Trisect AB at *h* and *k*. At *h* draw *ehf* perpendicular to AB meeting the arc BAD in *f* and the arc BAC in *e*. With centre A radius A*f* describe a circle giving the points *g*, *l*, *m*, *n*.

Bisect the arcs *fg*, *lm*, *nc* at *p*, *q*, *r*.

gl subtends 39° 35′ 38″·6 at A, as do *ef* and *mn*; while *fp*, *pg*, *lq*, *qm*, *nr*, *re* subtend 40° 12′ 10″·7 at A.

The construction given by Schwenter for inscribing a regular nonagon consists in taking half the side of the inscribed square as the side of the nonagon. The angle subtended by such a chord at the centre is 41° 24′ 34″.

The following approximations are also given by Schwenter :—

For the side of the regular 11-gon take 9/32 of the diameter.
„ „ „ 13-gon „ 1/4 „ „
„ „ „ 17-gon „ 11/60 „ „
„ „ „ 19-gon „ 1/6 „ „
„ „ „ 18-gon „ 1/5 of the side of the equilateral triangle.

§ 7. The following methods are given in Le Clerc's *Pratique de la Géométrie sur le Papier et sur le Terrain* (1668).

On a given straight line to describe any polygon from a hexagon to a dodecagon (fig. 14).

Let AB be the given straight line. Bisect it at O and on BA describe an equilateral triangle BAC. Divide the arc AC into six equal parts at M, N, P, Q, R. With C as centre and radii CM, CN, CP, CQ, CR describe arcs cutting OC produced in D, E, F etc. Assume D as the centre of the regular heptagon on AB, E as the centre of the regular octagon and so on.

Taking AB as unity this construction gives for the radius of the inscribed circle of the regular figure on AB

of 7 sides the value $\frac{\sqrt{3}}{2}+2\sin 5°$

8 „ „ $\frac{\sqrt{3}}{2}+2\sin 10°$

. . . .

12 „ „ $\frac{\sqrt{3}}{2}+2\sin 30°$

and from this the following table results :—

No. of Sides.	Approximate Value.	True Value of Radius.
7	1·0403368	1·0382608
8	1·2133218	1·2071068
9	1·3836634	1·3737387
10	1·5500656	1·5388418
11	1·7112620	1·7028440
12	1·8660254	1·8660254

The method can be applied as follows to obtain the regular polygons of from 12 to 24 sides on the line AB (fig. 15).

Describe the equilateral triangle as before. Divide the arc AC into twelve equal portions. Take as many of these as with twelve added will give the number of sides of the required figure.

Taking three to get the quindecagon, describe from the point C with radius CE an arc cutting IC produced in O. With O as centre and OB as radius obtain the point F, which is the required centre.

The distance FI for the regular figure of n sides where $24 > n > 12$ is

$$\frac{\sqrt{3}}{2}+2\sin\frac{n-12}{72}\pi+\sqrt{\frac{1}{4}+\left(\frac{\sqrt{3}}{2}+2\sin\frac{n-12}{72}\pi\right)^2}$$

giving the following values of FI.

No. of Sides.	Approximate Value.	True Value.
13	2·030	2·0281
14	2·194	2·1905
15	2·360	2·3523
16	2·526	2·5126
17	2·691	2·6748
18	2·855	2·8356
19	3·017	2·9950
20	3·179	3·1569
21	3·337	3·3173
22	3·494	3·4775
23	3·647	3·6340
24	3·798	3·7979

§ 8. A method of construction of the regular nonagon is given by Le Clerc. It consists, (fig. 16), in taking AB, AC, AD three radii joining the centre to consecutive angular points of an inscribed hexagon. CD is then produced, and on it an equilateral triangle, EGF having one vertex at E and a side equal to AB, is described; AG cuts the arc DB in H, the assumed vertex of a regular nonagon. The angle DAH is 39° 53′ 46″.

§ 9. The following approximation to the regular hendecagon is given by Le Clerc (fig. 17).

Take a radius of a circle AB. Bisect it in C, and on AC describe the equilateral triangle ACD. Mark off the chord AL equal to AC. With centre L and radius LD describe an arc cutting the circumference in O. Then CO is approximately the side of the hendecagon. The angle subtended by CO at the centre is 32° 44′ 29″ instead of 32° 43′ 38″. It is to be noted that the angle which AO subtends at B is 21° 12′ 42″. The angle which the side of the regular 17-sided figure would subtend at B is 21° 10′ 35″.

The calculation of this result suggested the following approximation. The length of the side of the 11-gon is equal to one-fifth the diagonal of the circumscribing square.

This gives an angle 32° 51′ 35″.

On the history and degree of certain geometrical approximations.

Part II.*

By A. J. Pressland, M.A.

§ 1. Since the former paper on this subject was read, Prof. Cantor has published the second volume of his history of Mathematics. This has necessitated various additions to the paper, which can perhaps be best given as an appendix.

On page 413 Prof. Cantor says that the construction of Dürer's pentagon is found in a book called *Geometria deutsch*, which was lately discovered in the town library at Nürnberg, and gives 1487 as the upper limit to its date. The construction is said to be "mit unverrücktem Zirckel," the same expression that Schwenter applies to Dürer's solution.

Lionardo da Vinci (1452-1519) gave several methods of accurate and of approximate construction. Thus in fig. 18.
the arc *ba* = 1/6, the arc *bc* = 1/3, the arc *cf* = 1/8, and the arc *af* = 1/24 of the circumference (p. 271).

Two constructions for the pentagon are also given by Lionardo. In fig. 19 the arcs are all of the same radius and the arc *am* is approximately 1/5 of the circumference, the value on calculation being found to be 72° 25′ (p. 272).

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* This was read at the Sixth Meeting, 8th April, 1892.

The second approximate solution consists in drawing an equilateral triangle ABC, and the perpendicular AD on BC. The side AC is assumed the chord of the regular pentagon in a circle radius DA. The angle so subtended is 70° 31′ 42″.

The following construction for the nonagon is due to Lionardo, (fig. 20). Let bna be an equilateral triangle; make $ng = ah$. Describe a circle with g as centre, radius ga. As the angle $bga = bka = 40^\circ\ 12'\ 28''$, the line ba is nearly the chord of the regular nonagon in this circle.

The expressions for the side of the 11-gon and 13-gon given by Schwenter are to be found in Dürer.

§ 2. The construction given by Abû'l Wafâ for the regular heptagon is known as the Indian rule. It has been applied in different forms, as above to the 11-gon and 13-gon, to the construction of regular figures.

It admits, however, of various modifications.

(i) Considering it under the following aspect, "the n^{th} part of the regular s-gon is the side of the regular p-gon in the same circle," the following results can be obtained:—

$s_{11} = 9/32$ of the diameter gives an angle 3′ 26″ too small.
$s_{13} = 1/4$ " " gives an angle 1° 15′ too large.

These are due to Dürer.

$s_{17} = 11/60$ of the diameter gives an angle 20° 7′ 40″·6.
instead of 21° 10′ 35″
$s_{18} = s_3/5$ gives an angle 19° 56′ 54″
instead of 20°
$s_{19} = 1/6$ of the diameter gives an angle 19° 11′ 17″·4
instead of 18° 56′ 41″.

These are due to Schwenter.

Beyond these the following may be worth mention:—

* $s_{29} = s_3/8$ giving an angle 12° 25′ 45″·2
instead of 12° 24′ 49″·6.
$s_{31} = s_4/7$ giving an angle 11° 34′ 43″
instead of 11° 36′ 46″.
$s_5 = 6s_{32}$ giving an angle 72° 2′ 42″
instead of 72° 0′ 0″.

This might be written $s_5 = 3s_{16}$ but the approximation is not so near.

$s_{25} = s_6/4$ giving an angle 14° 21′ 42″
instead of 14° 24′ 0.

(ii.) Considering the rule generalised thus, "The side of the *n*-gon minus the side of the *s*-gon is the side of the *r*-gon," the following results are found.

$s_3 - s_7 = s_7$
* $s_{10} - s_{17} = s_{25}$ giving 14° 23′ 32″·4
instead of 14° 24′ 0.
$s_7 - s_{16} = s_{13}$ giving 27° 38′ 6″
instead of 27° 41′ 32″.
* $s_{24} + s_{34} = s_{14}$ giving 25° 42′ 54″
instead of 25° 42′ 51″·4.

Formed on the analogy of
$s_5^2 = s_6^2 + s_{10}^2$ (Euclid XIII. 10)
and
$s_3^2 = s_6^2 + s_4^2$
are
* $s_{11}^2 = s_8^2 - s_{12}^2$ giving 32° 44′ 44″
instead of 32° 43′ 38″.
$s_{13}^2 = s_{12}^2 - s_{32}^2$ giving 27° 50′ 16″
instead of 27° 41′ 32″.
$s_{13}^2 = s_{10}^2 - s_{16}^2$ giving 27° 44′ 6″
instead of 27° 41′ 32″.

And the rougher approximation
$s_{11}^2 - s_{13}^2 = s_{21}^2$.

(iii.) Considering the rule generalised thus, "If AB be the side of the *n*-gon in a circle radius CA, and CA be bisected in D, then BD is the side of the regular *s*-gon," the following approximations may be obtained :—

s_6 gives s_7
s_8 gives for $2s_{17}$ the angle 21° 13′ 46″
instead of 21° 10′ 35″.
s_{10} gives for $2s_{19}$ the angle 19° 6′ 46″
instead of 18° 56′ 51″.

s_{12} gives for s_{10} the angle $36^\circ\ 6'\ 8''$
instead of $36^\circ\ 0'\ 0''$.
s_{15} gives for $2s_5$ the angle $70^\circ\ 54'\ 26''$
instead of $72^\circ\ 0'\ 0''$.
*s_{17} gives for s_{11} the angle $32^\circ\ 43'\ 44''\cdot8$
instead of $32^\circ\ 43'\ 38''\cdot2$,

an approximation first suggested by Le Clerc's construction for s_{11}.

s_{39} gives for s_{12} the angle $30^\circ\ 12'\ 48''$
instead of $30^\circ\ 0'\ 0''$

suggesting an approximation to s_{39}.

It is to be noted that this variation of the Indian rule is used in the Almagest construction (accurate) of the regular pentagon by Ptolemy.

(iv.) The following construction, though not an adaptation of the Indian rule, was suggested by it (fig. 21).

Let BE be the side of the n-gon in a circle radius OB. Bisect OB at D. Join ED and draw OF perpendicular to it. The locus of F is a circle with OD diameter. The angle DOF may give approximations to the side of the regular polygons.

Thus

with s_6 the angle DOF $= 0^\circ$.
with s_7 the angle DOF $= 8^\circ\ 58'$, giving an approximation to s_{40}
with s_8 the angle $= 16^\circ\ 19'\ 30''$,, ,, to s_{11}
*with s_9 the angle $= 22^\circ\ 29'$,, ,, to s_{16}
with s_{10} the angle $= 27^\circ\ 44'$,, ,, to s_{13}.

By reversing the construction of the fourth of these an extremely near approximation to s_9 is obtained, the angle being $39^\circ\ 59'\ 15''\cdot5$ instead of 40°.

§ 3. From the following construction certain approximations result.

Take a radius AB and draw the tangent at B. Take a point C on the circle and draw CE perpendicular to the tangent at B. Then $\sin CAB = \tan EAB$. Calling CAB θ, and EAB ϕ, this gives:

$$\theta = 40^\circ\,;\ \phi = 32^\circ\ 43'\ 56''\cdot7.$$

*From which s_9 gives s_{11}.

With the previous approximation to s_9 the angle obtained is $32^\circ\ 43'\ 32''\cdot5$. The true angle is $32^\circ\ 43'\ 38''\cdot1$.

The equation $\frac{1}{2}\sin\theta = \tan\psi$ gives:

$\theta = 40^\circ$; $\psi = 18^\circ\ 50'\ 58''\cdot5$, thus approximating to s_{19}.
With $\theta = 20^\circ$; $\phi = 18^\circ\ 52'\ 54''\cdot1$, again approximating to s_{19},

where the angle is $18^\circ\ 56'\ 41''$.

With $\theta = \dfrac{360^\circ}{19}$; $\phi = 17^\circ\ 59'\ 19''$. From which s_{19} gives s_{20}.

As s_{20} can be found exactly, this construction can be reversed.

With $\theta = 18^\circ$; $\phi = 17^\circ\ 10'\ 19''\cdot3$. From which s_{20} gives s_{21} the correct angle for s_{21} being $17^\circ\ 8'\ 34''$.

§ 4. In fig. 22, AB, CD are two diameters of a circle perpendicular to each other; AE, BF, tangents at A, B, are equal to four times the radius and the radius respectively. Join EF cutting the circle at M, N; and join AM, AN, cutting CD at m, n. Through m, n draw parallels to AB, namely GH, IK. The pentagon CIHGK is regular. M. Henri Barral, in *Nouvelles Annales*, XI. 388-390 (1852).

The construction above is given by Herr Staudt without proof in Crelle XXIV. (1842).

Terquem in a note says, "The construction of Herr Staudt is remarkable because it indicates an analogous construction for the division of the circumference into 17 equal parts." See also *Nouvelles Annales*, XVI. 310 (1857).

Among the calculations made for this paper the following occurred:—

$$61\cdot5 - 10\sqrt{5} = 39\cdot139320225,$$

a near approximation to the length 39·13929 ... inches of the seconds pendulum in London.

On Electrolysis.

By Professor MORRISON.

Third Meeting, January 8, 1892.

JOHN ALISON, Esq., M.A., F.R.S.E., Vice-President, in the Chair.

On the smallest number of entries necessary in a table of logarithms to seven decimal places.

By Professor J. E. A. STEGGALL.

Taking seven figure logarithms it is required to find at what interval from n, a number of five figures, the next entry need be made, in order that any intermediate logarithm may be calculated by the method of proportional parts.

We have

$$\log(n+d) - \log n = \mu\left\{\frac{d}{n} - \frac{d^2}{2n^2} + \ldots\right\} \quad \ldots \quad \ldots \quad (1),$$

and

$$\log(n+x) - \log n = \mu\left\{\frac{x}{n} - \frac{x^2}{2n^2} + \ldots\right\} \quad \ldots \quad \ldots \quad (2),$$

also

$$\frac{x}{d}\{\log(n+d) - \log n\} = \mu\left\{\frac{x}{n} - \frac{xd}{2n^2} + \ldots\right\}.$$

This latter quantity, which we add to $\log n$ in order to find $\log(n+x)$ approximately, is got by calculation, and its error (in defect) from the true quantity, (2), is

$$\mu x\left\{\frac{(d-x)}{2n^2} - \frac{(d^2-x^2)}{3n^3} + \ldots\right\};$$

the greatest value of this error is when $x = \frac{1}{2}d$, and it then is $\mu d^2/8n^2$.

In order that there may not be an error of a unit in the seventh place we must have $\mu d^2/8n^2 < \cdot0000001$.

Whence $$d/n < \sqrt{\cdot00000185}$$
or $$d < \cdot00136n,$$

and varies from 13·6 to 136 in the ordinary course of the tables.

For example, taking one of the most unfavourable cases

$$\log\ 99864 = 4{\cdot}9994090$$
$$\log 100000 = 5{\cdot}0000000$$

and therefore $$\log\ 99932 = 4{\cdot}9997045$$

which is correct to within a unit in the last figure.

In fact, the number of entries really required in a table is given by $$(1{\cdot}00136)^n = 10,$$

whence $$n = 1/{\cdot}00059024$$
$$= \text{about } 1700,$$

instead of 90000.

The table might be arranged

10000	to 20000	every	tenth	number	...	1000 entries
20000	„ 30000	...	twentieth	...	...	500 „
30000	„ 40000	...	fortieth	...	...	250 „
40000	„ 80000	...	fiftieth	...	...	800 „
80000	„ 100000	...	hundredth	...	...	200 „
						2750 „

If we apply the same method to a four figure table, all we require is that

$$\mu d^2/8n^2 < .0001,$$

or $$d < \sqrt{.0019n},$$
$$d < \ \ .044\, n.$$

Thus to obtain any logarithm to 4 decimal places we may, for example, work with only the logarithms of

104
108
112
116
120
125
130
135
140
145
150
155
160
165
170
175
180
185
190
195
200
208
216
224
232
240
250
260
270
280
290
300
312
324
336
348
360
375
390
400
416
432
448
464
480
500
520
540
560
580
600
625
650
675
700
730
760
790
800
830
865
900
930
960

To reduce the error to a maximum of ·00005, we require about the number of entries following: that is, about 100 entries against about 60.

103
106
109
110
113
116
119
120
123
126
129
130
133
136
140
144
148
150
154
158
160
164
168
170
175
180
185
190
195
200
205
210
215
220
225
230
235
240
245
250
255
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265
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275
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320
325
330
340
350
360
370
380
390
400
410
420
430
440
450
460
470
480
490
500
515
530
545
560
575
590
600
615
630
645
660
680
700
720
740
760
780
800
820
840
860
880
900
920
940
960
980

In each case the smallest number of entries has not been taken but a series of numbers has been arranged so as to make the use of the proposed table rapid and simple. The smallest number of entries is given for the four figure table correct to ·0001 by the numbers

104
108
112
116
121
126
131
136
142
148
154
160
167
174
181
188
196
204
212
221
230
240
250
261
272
283
295
307
320
334
348
363
378
394
411
429
447
466
486
507
529
552
576
601
627
654
682
712
743
775
809
844
881
919
959

or 55 entries in all; but it is clear that these entries are not so convenient as the somewhat larger number selected above.

On the Contact-Property of the Eleven-point Conic.

By R. E. Allardice, M.A.

It is sometimes the case that geometrical theorems, which are usually enunciated as properties of the triangle or the quadrilateral, may be stated more succinctly, and in a form that better suggests generalisation, as properties of the complete quadrilateral. Thus the theorem that is usually stated in the following form, "The three straight lines that join the middle points of opposite sides and the middle points of diagonals of a quadrilateral are concurrent and bisect one another at the point of concurrence," may be enunciated more simply as a property of the straight lines that join the middle points of the three pairs of opposite sides of a complete four-point; and the latter form of enunciation suggests how the theorem may be generalised by projection. So too the figure that consists of four points, any one of which is the orthocentre of the triangle formed by the other three, is better described as a complete four-point in which opposite sides intersect at right angles, (which may be called an orthic four-point).

This method of describing the property of the orthocentre suggests a way of stating the fundamental property of the nine-point circle, namely, as follows:—

In an orthic four-point, the middle points of the six sides and the three diagonal points are concyclic. It is now natural to enquire what the corresponding theorem is, when a quadrilateral of the most general kind is substituted for an orthic quadrilateral. It is well known that the nine-point circle is the locus of the centre of a conic that passes through the vertices of the orthic four-point; and in Taylor's *Ancient and Modern Geometry of Conics* (p. 283) it is proved that the locus of the centre of a conic that circumscribes a given four-point is a conic, (the eleven-point conic) which passes through the middle points of the six sides of the four-point, through the three diagonal points, and through two definite points at infinity which correspond to the circular points at infinity, in the case of the nine-point circle.

The following is an analytical proof of the analogue of the theorem that the nine-point circle touches the inscribed and escribed circles:—

Let ABCD (fig. 23) be the four-point; to show that the eleven-point conic of ABCD touches the similar conics inscribed in the triangles ABC, ABD, etc.

Take AC and BD as axes of co-ordinates; then the equations to AB, BC, CD, DA may be written

$$px+ry-1=0, \qquad qx+ry-1=0,$$
$$px+sy-1=0, \qquad qx+sy-1=0.$$

The equation to a conic through A, B, C, D, is

$$(px+sy-1)(qx+ry-1)+kxy=0\,;$$

that is

$$pqx^2+(pr+qs+k)xy+rsy^2-(p+q)x-(r+s)y+1=0.$$

The centre is given by

$$\left.\begin{array}{r}2pqx+(pr+qs+k)y-(p+q)=0,\\ (pr+qs)x+2rsy-(r-s)=0,\end{array}\right\}$$

that is

$$\left.\begin{array}{l}2pqx+(pr+qs)y-(p+q)+ky=0,\\ (pr+qs)x+2rsy-(r+s)\;+kx=0.\end{array}\right\}$$

Hence the locus of the centre is

$$x[2pqx+(pr+qs)y-(p+q)]-y[(pr+qs)x+2rsy-(r+s)]=0$$

that is,

$$2pqx^2-2rsy^2-\;(p+q)x+\;(r+s)y=0\,;$$

or

$$4pqx^2-4rsy^2-2(p+q)x+2(r+s)y=0. \quad \ldots \quad (1)$$

This is the equation to the eleven-point conic.

The equation to a similar conic may be written

$$4pqx^2-4rsy^2+2lx+2my+2n=0. \quad \ldots \quad \ldots \quad (2)$$

The line of intersection of these two conics is

$$[l+(p+q)]x+[m-(r+s)]y+n=0.$$

To determine l, m, n we have the conditions that (2) touches $y=0$, $px+ry-1=0$ and $qx+ry-1=0$, (taking the conic to be inscribed in ABC).

Assume

$$4pqx^2-4rsy^2+2lx+2my+2n\equiv ax^2+2hxy+by^2+2gx+2fy+c\,;$$

then, $\quad a=4pq,\ b=-4rs,\ c=2n,\ f=m,\ g=l,\ h=0.$

Hence,

$$\begin{array}{ll} A=-\ 8rsn-m^2, & F=-4pqm,\\ B=\ \ 8pqn-l^2, & G=\ 4rsl,\\ C=-16pqrs, & H=\ lm. \end{array}$$

The condition that (2) touch $\lambda x+\mu y+\nu=0$ is

$$A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0.$$

For the three sides of ABC, we have

1° $\lambda = \nu = 0,\ \mu = 1, \qquad \therefore\ B = 0$;

2° $\lambda = p, \quad \mu = r,\ \nu = -1, \quad \therefore\ Ap^2 + C - 2Fr - 2Gp + 2Hpr = 0$;

3° $\lambda = q, \quad \mu = r,\ \nu = -1, \quad \therefore\ Aq^2 + C - 2Fr - 2Gq + 2Hqr = 0.$

Taking (2) and (3) we have

$$A(p^2 - q^2) - 2G(p-q) + 2Hr(p-q) = 0; \text{ and}$$
$$A(p^2q - pq^2) - C(p-q) + 2Fr(p-q) = 0.$$

Hence $\quad A(p-q) - 2G + 2Hr = 0$; and

$Apq - C + 2Fr = 0$; and also $B = 0$.

These conditions reduce to

$$l^2 - 8pqn = 0,\ -(m^2 + 8rsn) + 16rs - 8mr = 0$$
$$\text{and } 8s(p+q) - 4m(p+q) + 4sl - lm = 0. \quad \ldots \quad \ldots \quad (3)$$

Now we have to show that

$(l+p+q)x + (m-r-s)y + n = 0$ touches the conic (1).

Assume

$$4pqx^2 - 4rsy^2 - 2(p+q)x + 2(r+s)y$$
$$\equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c';$$

then

$$A' = -(r+s)^2, \quad B' = -(p+q)^2, \quad C' = -16pqrs,$$
$$F' = -4pq(r+s),\ G' = -4rs(p+q),\ H' = -(p+q)(r+s).$$

Hence we have to show that

$$K \equiv -(r+s)^2(l+p+q)^2 - (p+q)^2(m-r-s)^2 - 8pq(r+s)(m-r-s)n$$
$$-16pqrsn^2 - 8rs(p+q)(l+p+q)n - 2(p+q)(r+s)(l+p+q)(m-r-s) = 0.$$

Now

$$K \equiv -(r+s)^2[l^2 + 2l(p+q) + (p+q)^2] - (p+q)^2[m^2 - 2m(r+s) + (r+s)^2]$$
$$-16pqrsn^2 - 8pqmn(r+s) + 8pqn(r+s)^2 - 8rsln(p+q) - 8rsn(p+q)^2$$
$$-2(p+q)(r+s)[lm - l(r+s) + m(p+q) - (p+q)(r+s)]$$

$$\equiv -m^2(p+q)^2 - 16pqrsn^2 - 8pqmn(r+s) - 8rsln(p+q)$$
$$-8rsn(p+q)^2 - 2lm(p+q)(r+s)$$

Now $\quad (p+q)^2(8rsn + m^2) = (p+q)(2lmr - 8rls), \quad$ [from (3)],

Hence

$$K \equiv (p+q)(8rls - 2lmr) - 16pqrsn^2 - 8pqmn(r+s) - 8rsln(p+q)$$
$$-2lmr(p+q) - 2lms(p+q)$$

Again, $\quad 8rls(p+q) = 4rlm(p+q) - 4rsl^2 - rl^2m$

$$= 4rlm(p+q) - 32pqnrs - 8pqnmr$$

$$
\begin{aligned}
\therefore \mathrm{K} \equiv\ & 4rlm(p+q) - 32pqnrs - 8pqmnr - 2lmr(p+q) - 16pqrsn^2 \\
& - 8pqmn(r+s) - 8rsln(p+q) - 2lmr(p+q) - 2lms(p+q). \\
\equiv\ & 2s[-16pqnr - 8pqrn^2 - 4pqmn - (p+q)(rln + lm)] \\
\equiv\ & s[-4l^2r - 2l^2rn - l^2m - 2l(p+q)(rn+m)] \\
\equiv\ & -ls[4lr + 2lrn + lm + 2(p+q)(rn+m)].
\end{aligned}
$$

Now if we eliminate s between the equations (3) we get

$$4lr + 2lrn + lm + 2(p+q)(rn+m) = 0 \; ; \; \therefore \mathrm{K} = 0.$$

It may easily be proved that the eleven-point conic passes through the following eleven points :—the vertices of the diagonal triangle of ABCD, the middle points of the six sides of ABCD and two definite points at infinity, namely, the centres of the two parabolas that pass through A, B, C, D ; and by means of the equation to the eleven-point conic given above, various particular cases may be considered.

[*Note.*—Since writing the above, I have found references to the following papers by Professor Beltrami on the eleven-point conic :— *Intorno alle coniche di nove punti e ad alcune questioni che ne dipendono*, Bologna, Mem. Accad., II., 1862, pp. 361-365, and Bologna, Rendiconto 1862-63, pp. 82-85 ; *Sulle coniche di nove punti*, Napoli, Giorn. di Mat., I., 1863, pp. 109-118 ; *Estenzione allo spazio di tre dimensioni dei teoremi, relativi alle coniche di nove punti*, Napoli, Giorn. di Mat., I., 1863, pp. 208-217, 354-360.]

Deduction of the Thermodynamical Relations.

By WILLIAM PEDDIE, D.Sc.

In his *Theory of Heat*, Clerk-Maxwell gives a very elegant and simple geometrical proof of the four thermodynamical relations, and points out that his construction shows that the truth of any one is a necessary consequence of the truth of any other. The usual analytical proof of these relations can be made as simple as Maxwell's geometrical proof, and the fact that any one is a necessary consequence of any other becomes evident when it is considered that they are all deduced by a common process from identical transformations of one equation. The following method of proof seems to me to bring out more directly their necessary interdependence.

Let u, v, x, y, be four variables any two of which may be taken as independent. If we take x and y as the independent variables we get

$$dv = \frac{\delta v}{\delta x}dx + \frac{\delta v}{\delta y}dy$$

where dx and dy are subject to no restriction other than extreme smallness of magnitude. If we now impose the condition $u =$ a constant, this equation takes the form

$$\left(\frac{\delta v}{\delta x}\right)_u - \left(\frac{\delta v}{\delta x}\right)_y = \left(\frac{\delta v}{\delta y}\right)_x \left(\frac{\delta y}{\delta x}\right)_u,$$

where the suffixes denote the quantity which is kept constant.

Now take u and x as the independent variables and we get

$$\left(\frac{\delta v}{\delta x}\right)_y - \left(\frac{\delta v}{\delta x}\right)_u = \left(\frac{\delta v}{\delta u}\right)_x \left(\frac{\delta u}{\delta x}\right)_y$$

whence $$\left(\frac{\delta v}{\delta y}\right)_x \left(\frac{\delta y}{\delta x}\right)_u = -\left(\frac{\delta v}{\delta u}\right)_x \left(\frac{\delta u}{\delta x}\right)_y \quad \ldots \ldots \quad (1)$$

From (1) we get

$$\left(\frac{\delta v}{\delta y}\right)_x = \phi(x, y)\left(\frac{\delta u}{\delta x}\right)_y \quad \ldots \ldots \ldots \ldots \ldots \quad (2)$$

$$\left(\frac{\delta v}{\delta u}\right)_x = -\phi(x, y)\left(\frac{\delta y}{\delta x}\right)_u \quad \ldots \ldots \ldots \ldots \ldots \quad (3)$$

where ϕ is a function whose dimensions are $[vxu^{-1}y^{-1}]$ and are therefore zero (the simplest-case) when $[vx] = [uy]$. In particular, when this condition holds, ϕ may be a constant (again the simplest case) which we may call positive or negative unity since its absolute magnitude depends upon the (arbitrary) units in which v, u, x, and y, are measured.

Equations (2) and (3) then become

$$\left(\frac{\delta v}{\delta y}\right)_x = \pm\left(\frac{\delta u}{\delta x}\right)_y \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (4)$$

$$\left(\frac{\delta v}{\delta u}\right)_x = \mp\left(\frac{\delta y}{\delta x}\right)_u \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (5)$$

From these equations we see that interchange of two dependent, or of two independent, variables does not change the ± sign; while the sign must be reversed when we interchange a dependent and an independent variable. Hence, remembering this rule, we may consider (4) above as the type of all such equations.

Now if
$$\left(\frac{\delta v}{\delta y}\right)_x = \pm\left(\frac{\delta u}{\delta x}\right)_y$$

we have some function, w, of x and y such that
$$\left(\frac{\delta w}{\delta x}\right)_y = \pm v\,;\left(\frac{\delta w}{\delta y}\right)_x = \pm u$$

or
$$dw = \pm v dx \pm u dy.$$

In the case of thermodynamics, the quantities e (energy), ϕ (entropy), t (temperature, p (pressure), v (volume), satisfy the conditions $[t\phi] = [pv]$ and $de = td\phi - pdv$. Hence, putting, in (4), $v = t$, $y = v$, $x = \phi$, $u = p$, we get
$$\left(\frac{\delta t}{\delta v}\right)_\phi = -\left(\frac{\delta p}{\delta \phi}\right)_v.$$

Interchanging t and ϕ,
$$\left(\frac{\delta \phi}{\delta v}\right)_t = \left(\frac{\delta p}{\delta t}\right)_v$$

Interchanging now p and v,
$$\left(\frac{\delta \phi}{\delta p}\right)_t = -\left(\frac{\delta v}{\delta t}\right)_p$$

Interchanging now t and ϕ,
$$\left(\frac{\delta t}{\delta p}\right)_\phi = \left(\frac{\delta v}{\delta \phi}\right)_p.$$

The investigation shows that the relations between the quantities ϕ, p, v, t, are the simplest which can satisfy the general relation (1) connecting four quantities of which any two may be regarded as independent.

Fourth Meeting, February 12, 1892.

Professor J. E. A. Steggall, M.A., President, in the Chair.

On Lines of Magnetic Force.

By Professor J. E. A. Steggall.

In my laboratory we make great use of the method of tracing lines of force described in Glazebrook and Shaw's *Practical Physics.* The field is modified in various ways; for instance, the fixed magnets are sometimes placed so as to somewhat resemble in their disposition the field magnets of a dynamo: sometimes a circular piece of soft iron is placed in the field, and the effect of its induced magnetization examined. The students like these exercises, and the results (of which some are exhibited) are very beautiful. We use steel bars about 8 c.m. long for fixed magnets, and small rather heavy lozenge-shaped needles, about one to two c.m. long, as what I shall call pointers. When we wish to eliminate the directive effect of the earth's magnetism we use two boards, of which the upper can rotate easily on the lower; on the upper the paper, magnets, and pointer are placed, and it is rotated until the pointer lies due North and South: this is determined by the coincidence of the direction of a light aluminium wire, attached to the pointer at right angles to its magnetic axis, with a thread or line from East to West. In this case we know that the force of all the influencing magnets, as well as of the earth, is directed North and South. The process is somewhat laborious, but when applied to a single magnet the resulting lines of force agree with remarkable exactness with those given by theory for an indefinitely thin magnet.

In making these experiments it occurred to me to examine whether any graphic methods similar to those given by Lloyd and others could be applied to determine the theoretical form of the lines of force due to the joint influence of a bar magnet and the earth. A result of a very simple kind was arrived at in one case, namely, when the magnet lies in the magnetic meridian; and having drawn the curves of a bar magnet alone to scale, it was possible to deduce these for the same magnet lying as described, with the result that these lines also agreed most minutely with those determined by experiment. The method used can only be applied in the case cited.

Let A, B, denote the poles of a magnet in the earth's meridian, P any point, co-ordinates x, y, m the intensity of a pole, H the earth's horizontal force, $\mathrm{AP} = r_1$, $\mathrm{BP} = r_2$, angle $\mathrm{PAB} = \theta$, angle $\mathrm{PBA} = \pi - \theta_2$, then if s be the arc of a line of force, resolving normally,

$$\frac{m}{r_1^2} \cdot \frac{r_1 d\theta_1}{ds} - \frac{m}{r_2^2} \cdot \frac{r d\theta_2}{ds} + \mathrm{H} \cdot \frac{dy}{ds} = 0$$

and $$r_1 \sin\theta_1 = r_2 \sin\theta_2 = y,$$

therefore $$m \sin\theta_1 \frac{d\theta_1}{ds} - m \sin\theta_2 \frac{d\theta_2}{ds} - \mathrm{H} y \frac{dy}{ds} = 0$$

or $$m(\cos\theta_1 - \cos\theta_2) + \tfrac{1}{2}\mathrm{H} y^2 = \text{constant}$$

is the equation to the lines of force.

If the magnet is not along the meridian the equation cannot be integrated as it stands.

Now if we trace the systems of curves

$$m(\cos\theta_1 - \cos\theta_2) = \text{constant},$$
$$\tfrac{1}{2}\mathrm{H} y^2 = \text{constant},$$

of which the former represents the lines of force due to a magnet alone, the latter a series of parallel straight lines, and if the constants form arithmetical progressions of equal differences, the consecutive intersections give the lines of force due to the disposition of the magnet considered.

The numerical ratio of m and H can be at once obtained by measuring the distance of the axial points of equilibrium from the ends of the magnet: this gives, a, b being these distances

$$m\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \mathrm{H}\,.$$

In a similar way to the above equi-potential lines may be drawn by having a short aluminium lozenge fixed at right angles to the pointer.

Fifth Meeting, March 11, 1892.

JOHN ALISON, Esq., M.A. F.R.S.E., Vice-President, in the Chair.

On the relation between the stereographic projections of points of a plane related to one another by inversion.

By CHARLES TWEEDIE, M.A.

What is meant will best be understood by the following:—

1. Suppose we have an unlimited straight line XX′ (fig. 24) with a finite point O on it. Take two points on the line, C and C_1 in the same sense with respect to O such that $OC \cdot OC_1 = a^2$. Then C and C_1 are connected by inversion, and O is the centre inversion. Without loss of generality for what follows we may, for convenience, take

$$a^2 = 1^2 = 1,$$

so that

$$OC \cdot OC_1 = 1.$$

Then if we take two points on the line such that $OK = OK' = 1$, we obviously divide the straight line into two parts—KK′ and the unlimited part outside KK′—in such a way that to any point C outside KK′ there is one and only one corresponding point C_1 within KK′; likewise C_1 has only the corresponding point C.

2. Take now a circle with KK′ as diameter and DD′ as diameter perpendicular to KK′. Project on this circle from D the unlimited straight line XX′. To any point on the line there will be one and only one point on the circle. The part of XX′ outside KK′ projects into the semicircle KDK′, while KK′ projects into KD′K′.

Let γ be the projection of C, γ_1 that of C_1 the inverse of C. Join $O\gamma$ and $O\gamma_1$. Denote the angle ODC by ϕ, the angle ODC_1 by ϕ_1, the angle $KO\gamma$ by χ, and the angle $KO\gamma_1$ by χ_1.

Let $$OC = r,$$

then $$OC_1 = 1/r.$$

Then $$\tan\phi = r, \ \tan\phi_1 = 1/r$$

$$\therefore \qquad \phi + \phi_1 = \pi/2,$$

$$\therefore \qquad D'O\gamma + D'O\gamma_1 = 2\phi + 2\phi_1 = \pi,$$

that is, $$\pi/2 + \chi + \pi/2 - \chi_1 = \pi,$$

$$\therefore \qquad \chi = \chi_1;$$

that is, γ_1 is the reflection in XX′ of γ; and now from the obvious symmetry we see that, given any point C, to find its inverse C_1 all we have to do is to find γ by projection and then join D′ to γ, cutting OX in C_1.

Or we may say, if γ is characterised by χ on the circle, then γ_1 is characterised by $-\chi$.

3. Take now the figure so obtained, and with DD′ as axis give it a complete revolution (fig. 25).

XX′ then generates a plane, the circle KDK′ a sphere; KK′ generates a circular area $K\gamma K'\gamma'$ in the plane, such that to any point of the plane outside it $C(r, \theta)$ there corresponds one inverse point $C_1(1/r, \theta)$ within it, and *vice versa.*

C and C_1 project into γ and γ_1 on the sphere which are now characterised by (θ, χ) and $(\theta, -\chi)$, so that the one is the reflection, in the plane, of the other. And if C trace out any curve in the plane, while C_1 traces out its inverse, then γ will trace out a curve Γ on the sphere, while γ_1 will trace out Γ_1—its reflection in the plane, and therefore an exactly similar though not congruent curve.

From obvious symmetry it is also possible to obtain Γ and Γ_1 from the trace of C by projecting it respectively with respect to D and to D′.

II.

4. Take now the more important case, and closely connected with the former, of points of an Argand plane (fig. 26) connected by the relation $zz_1 = 1$, z and z_1 being the representatives of the two points C and C_2 so related.

5. Writing z in the form

$z = r(\cos\theta + i\sin\theta)$, ($r$ and θ being the polar co-ordinate of C),

we have $$z_1 = 1/r(\cos\theta + i\sin\theta) = (\cos\theta - i\sin\theta)/r;$$

and hence the polar co-ordinates of C_2 are $(1/r, -\theta)$.

Therefore for every point C outside the circle $K\gamma K'$ we have a point C_2 within it and only one. In fact, C_2 is simply the reflection in XX' of C_1 the inverse of C.

Moreover, since C_1 and C_2 are obviously situated on the same circle round O_1 when we project on the same sphere as before the χ_1 of C_1 and the χ_2 of C_2 are the same; so that if (θ, χ) characterise γ the projection of C on the sphere, then $(-\theta, -\chi)$ will characterise γ_2 the projection of C_2.

Therefore γ_2 is the reflection in the straight line XX' of γ_1, and it is only in particular cases of symmetry about XX' of the traces of C and C_2 that the curves Γ and Γ_2 will, taken as a whole, be the reflections of each other in the plane.

But in another sense the symmetry with regard to D and D' is greater. Looking on $D\gamma$ and $D'\gamma_2$ as indexes tracing out Γ and Γ_2, we see that if $D\gamma$ start to move tracing out any curve Γ, then the motion of $D'\gamma_2$ looking at it from D' is exactly the same in every respect in the tracing out of Γ_2.

6. The case of Inversion may be looked on as a correspondence of points z' and z in an Argand plane through the relation $z'\bar{z} = 1$, where $\bar{z}$ is the conjugate of z.

For
$$\begin{aligned} z' = 1/\bar{z} &= 1/r(\cos\theta - i\sin\theta) \\ &= (\cos\theta + i\sin\theta)/r. \end{aligned}$$

so that the polar co-ordinates of z' and z are $(1/r, \theta)$ and (r, θ) respectively.

7. From the projection on the sphere it is easy to see what curves, taken as a whole, are transformed into themselves. The simplest case is that of the circle, (and it is assumed that circles of the plane project into circles of the sphere, and *vice versa*).

In the first case, that of Inversion, the only circles which reflect themselves in the plane are those which cut orthogonally the great circle in which the plane meets the sphere. Their projections in the plane therefore are circles obviously cutting the equatorial circle orthogonally. It is equally obvious that the latter circle goes over point for point into itself.

In the second case, the circles must mirror themselves in XX'.

There are two sets of circles which do this :—

1°. All great circles passing through K and K'; and 2°. all circles of the sphere, whose planes are perpendicular to XX', and therefore cutting the first set, including the equatorial circle orthogonally.

In Projection, the first set goes into circles passing through K and K′; the second set into circles orthogonal to the circle $K\gamma K'$, and whose centres obviously lie on XX′. The two sets being orthogonal on the sphere, their projections on the plane may also be shown to be orthogonal. The only two points which go into themselves are the two K and K′.

These results are thus simply read off from the appearance of the projections on the sphere.

8. We might look on the whole operationally as follows :—

Let V operating on z or $V(z)$ produce $1/z$. Let W denote the corresponding operation on the corresponding point γ on the sphere, so that $W(\gamma)$ denotes a mirroring with respect to the straight line XX′ of γ or $W(\gamma) = \gamma_2$.

Then $$V^2(z) = V(V)(z) = V(1/z) = z$$

∴ $$V^2 = 1\,;$$

and obviously $$W^2(\gamma) = W(\gamma_2) = \gamma,$$

since it is simply reflected back,

∴ $$W^2 = 1.$$

Similarly let $\overline{V}$ denote the operation on z

$$\overline{V}(z) = \overline{(1/z)} = 1/(x - yi) = 1/r(\cos\theta - i\sin\theta)$$

corresponding to Inversion, and W the corresponding operation on γ of the sphere, *i.e.*, reflection of γ in the equatorial plane into γ_2.

Then $$\overline{V}^2(z) = \overline{V}(\overline{V}(z)) = \overline{V}(1/z) = z$$

∴ $$\overline{V}^2 = 1,$$

and similarly $$\overline{W}^2 = 1.$$

9. For example; to find the points which go into themselves in Inversion, we have to put $\overline{V}(z) = z$

or $$1/(x - yi) = x + yi$$

∴ $$x^2 + y^2 = 1$$

i.e., every point in the equatorial circle goes over into itself as the operation W obviously allows.

For the second case we have to put

$$V(z) = z \quad i.e., \ (1/z) = z$$

$$\therefore \quad x^2 - y^2 + 2xyi = 1$$

$$\therefore \quad xy = 0 \text{ and } x^2 - y^2 = 1$$

with the solutions

$$(1)\ x = 0 \qquad y = \pm i$$

$$(2)\ y = 0 \qquad x = \pm 1.$$

The 1° solution on application of W is easily shown to be false, while the 2° solution, corresponding to K and K′, satisfies the conditions.

Experimental Introduction to the Study of Magnetism.

By Professor C. G. Knott, D.Sc., F.R.S.E.

The aim of the scientific teacher is to teach the pupil how to think along scientific lines. By a suitable presentation of the facts of experience he should lead the mind of the learner to form almost intuitively the scientific law or generalisation which embraces them all. We may of course start with the law or formula, and develope it mathematically into all its ramifications. But that reduces itself to mere analytical skill. If carried out faithfully in the elementary teaching of science, it would tend to give the learner an erroneous conception of the whole method of scientific investigation and the meaning of scientific law. On the other hand, if we simply present a series of curious experiments without distinct intellectual linkage, we place the learner in the position somewhat of the old lady who, after reading through the dictionary, remarked that "it was vera interestin' readin', but a wee disconnecked."

In what follows I propose to indicate a course on magnetism which seems to lead naturally to a scientific grasp of the fundamental principles of the science. At the outset I may state briefly what seem to be the faults of the courses usually given in books, and presumably in schools and colleges.

First, from the very outset, a magnet is regarded as made up of two parts. Now there is a great advantage in certain problems of a more advanced character so to regard the magnet. But, to introduce the idea of *separate* poles at the very beginning, and to appeal to electrostatics for an analogy, is (unless the comparison be most carefully qualified) to suggest to the mind *separable* poles, and to raise up a needless barrier to the proper appreciation of an electromagnetic shell when the time comes—and it ought soon to come—for the discussion of iron-less magnets. A magnet is a whole, and should be treated as such.

Secondly, the conception of *poles* as *points* where magnetism is concentrated is made far too much of. Unless the statements are safeguarded by qualifications, inconsistency, if not confusion, is almost certain to appear sooner or later. For example, on page 4 of Mr Poyser's *Magnetism and Electricity*, a diagram is given showing the neutral line and magnetic axis of a bar magnet. The axis is defined as "the line joining the poles;" the poles are defined as "two points where the attractive power is greatest." Yet if we glance at the experimental figure on page 3, we shall see that the greatest "attractive power" is at the corners, which of course do not lie on the axis. The only true experimental definition of a pole is that *region*, not necessarily and not usually a *point*, towards which the lines of force converge. By using the word in this broad sense, we find no difficulty in at once appreciating the magnetic character of, say, a galvanometer coil, which certainly cannot for a moment be regarded as having point poles.

Thirdly, the important truth that the metal part of a bar magnet is not the whole magnet is not emphasised. The metal part is the nucleus; and the associated magnetic field is as much part of the magnet as the steel or iron. The conception of the magnetic field should be introduced as soon as possible.

Fourthly, the physical quantity by which we measure the strength or power of a given magnet is introduced as a product of two quantities, neither of which can be measured. Almost all books define magnetic moment as the product of the strength of the magnetic pole into the distance separating the poles; the books that do not so define it make no mention of it. And yet magnetic moment is *the* quantity with which we have to deal. It is the basis of our whole system of magnetic and electromagnetic measurements. It has long seemed to me that experiments should be so chosen as to bring out this property directly, and impress it upon the mind. What I now proceed to give is essentially a series of lecture notes, which requires amplification at the hands of the teacher. To make the course complete in itself, I must include many of the familiar experiments; but the manner in which these are arranged and discussed in connection with new, or at least unfamiliar, experiments is sufficiently original to merit, I think, some little attention. I assume the reader's ability to supply all the usual definitions of the terms used.

I. The orientation of a horizontal magnet freely suspended; in fact, the well-known property of the mariner's compass.

Mark the ends; always the same end which points north.

Do the same for a second magnet; and establish the usual laws of apparent attraction and repulsion between the dissimilarly and similarly marked ends.

This principle enables us rapidly to mark magnets.

II. Study positions of a magnet in which it does not disturb the lie of a suspended needle; both being in the same plane.

Determination of certain critical configurations, in which the needle seems to lose its power of pointing north. Why? Because the magnet's influence counteracts the original influence.

Conclusion: the earth is a magnet.

Introduce the conception of a magnetic Field or region where a free magnet points in a definite direction. The behaviour of a magnet in a magnetic field demonstrates the existence of both magnet and field. Every magnet has a field associated with it.

III. Induction the fundamental action of the magnetic field.

Usual experiments with nails, iron filings, etc. The iron first magnetised, and then acted upon as if it were a magnet. Study of the configuration of a field near a magnet by the help of iron filings.

IV. Experimental tracing of lines of force by step by step motion of a small magnet. The combined field of force due to a magnet and the earth's horizontal field studied very easily by this means (see Glazebrook's *Practical Physics*).

To study the lines of force due to a magnet only, suspend the small exploring magnet over a loose sheet of paper, and render it astatic by a large magnet suitably adjusted. Lay the magnet whose field is to be studied on a marked position on the paper. Then by moving the sheet with magnet attached under the suspended magnet we may point by point trace out a line of force. For any new position the sheet must be adjusted, so that the one end of the suspended magnet lies exactly over the point which, in the immediately preceding position, lay under the other end.

The above might not be suitable for showing in a lecture. In this case it will suffice to show, by study of the behaviour of a suspended needle in the presence of any magnet, what are the directions of the field in the axial line (end-on), and in what might be called the equatorial plane (broadside-on) of the magnet.

V. Quantitative properties of magnetic fields associated with magnets.

Many modes of attack.

The following seems to have special merits :—

Suspend a needle in the earth's field, and note its position.

Set a magnet end on to east, and adjust till the needle is deflected through 45°. We see at once that the needle is under the action of two equal fields (the earth's and the magnet's); for they are at right angles to each other, and make equal angles with the needle on opposite sides of it. Measure the distance between the middle points of magnet and needle; take number of vibrations of the needle, say in five minutes.

Invert the magnet and adjust again till the deflection is 45°. Measure the distance as before; and take vibrations.

Place the magnet to the west, and make exactly the same adjustments for the two end-on positions of it; each time taking vibrations.

Take the mean of the four distances so obtained; and at this distance set the magnet on the north and south line, first to the north and then to the south. Take the vibrations in all positions; for only two will this be possible. For the other two the needle is nearly astatic.

Take the magnet right away; and count the vibrations of the needle when under the earth's influence only.

Let n_0 = number of vibrations per minute when earth's field (H) acts alone.

n_1 = mean of numbers of vibrations in the four east and west positions of the magnet.

n_2 = mean of numbers of vibrations in the two possible north and south positions, in which the field is obviously 2H.

Then it will be found that

$$n_0^2 : n_1^2 : n_2^2 = 1 : \sqrt{2} : 2.$$

May use two magnets, and get n_3^2 proportional to 3H.
Thus we find

1. Mutually perpendicular equal fields are compounded like forces.

Easy to extend this for all fields by showing that two magnets, each giving 45° when acting singly, will give when acting together (one to the east and one to the west) a deflection of 63½°, whose tangent is 2.

2. When a given magnet vibrates in a field, the square of the number of vibrations per second varies as the strength of the field.

This vibration law is of supreme importance. As in the case of the pendulum, the acting couple is proportional to the square of the number of vibrations; and thus we conclude that the couple acting on a magnet is proportional to the field, other things being the same.

VI. Comparison of Magnetic Moments. The simplest way to define the magnetic moment of a magnet is to say that it is proportional to the couple acting on it when it is set east and west in the earth's horizontal field.

We could then express all magnetic moments in terms of the magnetic moment of a chosen standard magnet.

The couples may be measured statically. More accurately, however, by small oscillations.

The moment of inertia of the magnets may be eliminated thus. Take three magnets of moments m_1 m_2 m_3, all unknown. Suspend all by a light frame at the end of a long thin wire. Let τ be the rotational moment brought into play in virtue of the torsion of the wire. Have all three magnets pointing similarly, and take vibrations. Invert each in turn, and take vibrations for each configuration. Then if n_0 n_1 n_2 n_3 are the vibration numbers, we get

$$\begin{aligned} m_1 + m_2 + m_3 + \tau &\propto n_0^2 \\ -m_1 + m_2 + m_3 + \tau &\propto n_1^2 \\ +m_1 - m_2 + m_3 + \tau &\propto n_2^2 \\ +m_1 + m_2 - m_3 + \tau &\propto n_3^2. \end{aligned}$$

Hence

$$m_1 : m_2 : m_3 : \tau$$
$$= n_0^2 - n_1^2 : n_0^2 - n_2^2 : n_0^2 - n_3^2 : (n_1^2 + n_2^2 + n_3^2 - n_0^2)/2$$

In this way the significance of magnetic moment is made obvious.

I have made the experiment, and found it very satisfactory.

VII. By the familiar Gaussian experiments, we readily establish that in the axial and equatorial directions from a given magnet

$$\text{Field} \propto 1/(\text{Distance})^3$$

if the distance is not too small.

Also that the equatorial field is half the axial field at the same distance from the magnet's centre.

Also that at equal distances from different magnets, the fields are proportional to the magnetic moments as given in VI. Hence we may put

$$\begin{aligned} \text{Axial} \quad \text{Field} &= 2\text{K}/r^3 \\ \text{Equatorial ,,} \quad &= \text{K}/r^3 \end{aligned}$$

where K is the magnetic moment measured in chosen units.

Note.—This law is exact for spherical magnets; and might be established by using electromagnets with spherical or short cylindrical iron cones.

It is best to use the earth's field as provisional unit.

VIII. Measurement of H and K in absolute units.

The couple vanishes when the magnet points north or south in the earth's field, and is at a maximum when it points east and west.

This maximum couple is proportional to the field and to the magnetic moment.

Define it as being measured by the product of the two—say KH.

We may suppose this measured statically in absolute units.

Thus HK = C a known quantity.

By deflection experiments, end-on, at distance r

$$\frac{2K}{r^3}/H = \tan\theta$$

or $$K/H = \tfrac{1}{2}r^3\tan\theta, \text{ a known quantity.}$$

Hence $$\left.\begin{aligned} K^2 &= \tfrac{1}{2}Cr^3\tan\theta \\ H^2 &= \frac{2C}{r^3\tan\theta} \end{aligned}\right\}$$

and H and K are both found.

IX. Potential Energy of Magnet in Field.

For any deflection θ, it may be shown, by suitable combinations as in Section V., that the couple acting on the magnet K in field H is $HK\sin\theta$.

Hence work done in moving the magnet from θ to θ' is

$$\begin{aligned} \int_{\theta}^{\theta'} HK\sin\theta d\theta &= -HK(\cos\theta' - \cos\theta) \\ &= V' - V \end{aligned}$$

where V′ and V are the potential energies in the positions θ' and θ. We may write

$$V = \text{constant} - HK\cos\theta.$$

Now in the earth's field there is no tendency to translation, so that the potential energy depends only on orientation.

This holds for all magnets in uniform fields; and may be assumed to hold for the magnetic molecule in any field.

Putting then the constant = 0, we get

$$V = -HK\cos\theta.$$

This makes the potential energy vanish when the magnet is at right angles to the lines of force.

X. Mutual action of two small Magnets.

From the fact that magnetic fields are superposable, we infer that a small magnet may be decomposed into any three elements, just as a force is decomposed.

Let K, K′, be the magnetic moments of two magnets making direction cosines $\lambda\ \mu\ \nu$, $\lambda'\ \mu'\ \nu'$, with regard to any chosen system of axes, of which one (the x-axis) is the line joining their centres.

Let r be the distance between these centres.

Decompose K into $K\lambda$, $K\mu$, $K\nu$,
and K′ „ $K'\lambda'$, $K'\mu'$, $K'\nu'$.

By application of IX. we readily find the potential energies of the different pairs of components. They are shown tabulated in the following scheme:—

TABLE OF POTENTIAL ENERGIES BETWEEN VARIOUS COMPONENTS.

	$K'\lambda'$	$K'\mu'$	$K'\mu'$
$K\lambda$	$-\frac{2KK'}{r^3}\lambda\lambda'$	0	0
$K\mu$	0	$+\frac{KK'}{r^3}\mu\mu'$	0
$K\nu$	0	0	$+\frac{KK'}{r^3}\nu\nu'$

Hence the potential energy of the two magnets is

$$W=\frac{KK'}{r^3}\{(\lambda\lambda'+\mu\mu'+\nu\nu')-3\lambda\lambda'\}$$

Or if θ, θ', are the angles made by K, K′, with the line joining them, and ϕ is the angle between K, K′,

$$W=\frac{KK'}{r^3}(\cos\phi-3\cos\theta\cos\theta').$$

If they lie in one plane

$$\phi=\theta'-\theta$$

and

$$W=\frac{KK'}{r^3}(\sin\theta\sin\theta'-2\cos\theta\cos\theta').$$

Apply this to study the field due to a given magnet; in other words, to find the direction and magnitude of the force at a point whose co-ordinates are r and θ referred to the centre and axis of the magnet.

Let K be the magnet of reference.

Let K$'$ be another small magnet placed at the point $(r\theta)$, and let K$'$ be placed perpendicular to the line of force so that W vanishes. This condition gives

$$\tan\theta'\tan\theta = 2.$$

Hence if β be the angle between r and the line of force

$$\tan\theta' = \cot\beta,$$

and hence $$\tan\theta = 2\tan\beta.$$

To find H the value of the force, let K$'$ lie along r; then $\theta' = 0$ and $$\mathrm{W} = -\mathrm{K}'2\mathrm{K}\cos\theta/r^3.$$

But this may also be written

$$\mathrm{W} = -\mathrm{K}'\mathrm{H}\cos\beta$$

$$= \quad \mathrm{K}'\mathrm{H}/\surd(1+\tfrac{1}{4}\tan^2\theta).$$

Hence $$\mathrm{H} = \frac{\mathrm{K}}{r^3}\surd(3\cos^3\theta+1).$$

XI. Case of circular coil.

Following out the same method, let us consider the field due to a circular coil at any point in its axis, assuming Ampère's rule that such a circuit is equivalent to any coterminous magnetic shell, magnetised normally with strength equal to the current.

Take as shell the cone with vertex at P; and place at P the magnet K pointing along the axis of the coil.

The element at S is equivalent to a small magnet of moment ωi (ω the small area) and with axis $\perp^r$ to PS.

Hence the mutual potential energy of each element and K is

$$w = \frac{i\omega\mathrm{K}}{\mathrm{PS}^3}(\sin 90^\circ\sin a - 2\cos 90^\circ\cos a)$$

$$= \frac{i\omega\mathrm{K}}{\mathrm{PS}^3}\sin a$$

where a is the semi-vertical angle at P.

Hence for the whole circular strip through S

$$dW = 2\pi K i r \sin^2 a dr/r^3$$
$$= 2\pi K i \sin^2 a dr/r^2$$

Integrating from $r = R$ to $r = \infty$, we get for the potential energy of the circuit and the magnet

$$W = -\frac{2i\pi K \sin^2 a}{R} = -K\frac{2\pi i a^2}{R^3}$$

where a is the radius of the circle.

Hence the field at the point P is

$$i\frac{2\pi a^2}{R^3}.$$

At the centre this becomes

$$i\frac{2\pi}{a}.$$

In this last example, it is assumed that the equivalence of circuits and magnets has been established experimentally. For this purpose the usual experiments are amply sufficient.

The experimental treatment of the subject of magnetic induction has been greatly improved in these later days, thanks chiefly to such men as Ewing and Hopkinson, following up along the lines of Faraday and Maxwell.

Sixth Meeting, April 8, 1892.

Professor J. E. A. Steggall, M.A., President, in the Chair.

On a surface of the third order.

By R. E. Allardice, M.A.

In the second part of Professor Chrystal's *Algebra* (Exer. V., No. 4) the following exercise is given :—

If $2xyz - x^2 - y^2 - z^2 + 1 = 0$, and x, y, z are all real, then all, or none, of the quantities x, y, z lie between -1 and $+1$.

This result follows at once from the fact that, if D be the x-discriminant of the above function, then

$$D \equiv 4(y^2z^2 - y^2 - z^2 + 1) \equiv 4(y^2 - 1)(z^2 - 1).$$

It occurred to me to consider the geometrical interpretation of this theorem, and then to consider the surface itself which the above equation represents; and finally I made a rough model in clay of the finite sheet of the surface, got this model cast in plaster, and exhibited the cast to the *Society*, giving at the same time some account of the properties of the surface. The following is a brief account of some of these properties:—

The geometrical interpretation of the theorem quoted above is, that part of the surface lies within the cube formed by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$; but that no part of the surface lies within any one of the six infinite spaces that can be generated by the motion of a face of the cube in a direction perpendicular to that face. From this it follows, that, if the surface is continuous, certain of the vertices of the cube must be conical points on the surface. It is easily seen that the points $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ lie on the surface; and they are therefore conical points. It may easily be shown that the enveloping cones at the conical points are right circular cones with their vertical angles right angles.

The section made by each of the co-ordinate planes is a circle (with a straight line at infinity).

The section made by a plane that bisects the angle between two of the co-ordinate planes consists of a parabola and a straight line (which is a diagonal of one of the faces of the cube referred to above).

All the sections parallel to one of the co-ordinate planes are ellipses and hyperbolas inscribed in the same square.

There are nine real straight lines on the surface; of which six are at a finite distance, and consist of one of the diagonals of each of the six faces of the cube; and the other three are at an infinite distance in the direction of the co-ordinate planes.

As four is the maximum number of conical points that a cubic surface may have, the surface under consideration possesses the maximum number. A cubic surface contains, in general, twenty-seven straight lines; but when a straight line passes through a conical point it counts for two, and when it passes through two conical points, for four. Thus, in the present case, the six straight lines at a finite distance, which are the six edges of a tetrahedron whose vertices are the four conical points, count for twenty-four. The remaining three straight lines always lie in one plane; in the case of the surface we are now considering, this plane is the plane at infinity.

The projection on the z-plane of the section made by any plane containing the conical points $(1, 1, 1)$ and $(-1, -1, 1)$ is a rectangular hyperbola whose asymptotes are parallel to the axes of x and y.

If we make the equation homogeneous, we get

$$2xyz - ax^2 - ay^2 - az^2 + a^3 = 0\,;$$

the finite sheet of which surface is contained by the planes $x = \pm a$, $y = \pm a$, $z = \pm a$. If we make a zero, the surface degenerates into the three co-ordinate planes.

The equation may be further generalised in the form

$$2abcxyz - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2 + a^2b^2c^2 = 0.$$

ILLUSTRATION OF THE COMPOSITION OF TWO SIMPLE HARMONIC MOTIONS.

The composition of two simple harmonic motions, of equal amplitudes, in directions at right angles to one another, may be illustrated by means of the surface $2xyz - ax^2 - ay^2 - az^2 + a^3 = 0$, in the case in which the periods of the motions are very nearly equal.

FIRST METHOD.

We may write

$$\begin{aligned} x &= a\cos\omega t, \\ y &= a\cos(\omega' t - \alpha) \\ &= a\cos\{\omega t + (\omega' - \omega)t - \alpha\} = a\cos(\omega t + \theta), \end{aligned}$$

where $\omega' - \omega$ is small, and hence θ, which is equal to $(\omega' - \omega)t - \alpha$, varies slowly with the time.

The resultant of these two simple harmonic motions for a small interval of time, is given by

$$x^2 + y^2 - 2xy\cos\theta = a^2\sin^2\theta. \qquad (1)$$

Multiplying this equation by a, and comparing with the equation to the surface, we have

$$2xya\cos\theta - ax^2 - ay^2 + a^3\sin^2\theta = 0\ ;$$
$$2xyz - ax^2 - ay^2 - az^2 + a^3 = 0\ ;$$
$$\therefore \qquad z = a\cos\theta \text{ and } -az^2 + a^3 = a^3\sin^2\theta,$$

and these two equations are consistent.

If we look upon (1) as the equation to a family of curves, it will always be possible to obtain them as the projection of a family of curves on any surface; but it will not in general be possible to obtain z as a function of the parameter θ alone.

SECOND METHOD.

In the equation

$$2xyz - ax^2 - ay^2 - az^2 + a^3 = 0,$$

put
$$x = a\cos\omega t, \quad y = a\cos(\omega t + \theta).$$

Then

$$2a^2z\cos\omega t\cos(\omega t + \theta) - a^3\cos^2\omega t - a^3\cos^2(\omega t + \theta) - az^2 + a^3 = 0\ ;$$

that is

$$z^2 - 2az\cos\omega t\cos(\omega t + \theta) + a^2\{\cos^2\omega t + \cos^2(\omega t + \theta) - 1\} = 0.$$

If D be the discriminant of this equation, we have

$$\begin{aligned} D/4a^2 &= \cos^2\omega t\cos^2(\omega t + \theta) - \{\cos^2\omega t - \cos^2(\omega t + \theta) + 1\} \\ &= (1 - \cos^2\omega t)(1 - \cos^2(\omega t + \theta)) = \sin^2\omega t\sin^2(\omega t + \theta). \end{aligned}$$

Hence
$$\begin{aligned} z &= \{2a\cos\omega t\cos(\omega t + \theta) \pm \sqrt{D}\}/2 \\ &= a\{\cos\omega t\cos(\omega t + \theta) \pm \sin\omega t\sin(\omega t + \theta)\} \\ &= a\cos\theta \quad \text{or} \quad a\cos(2\omega t + \theta). \end{aligned}$$

The second of these solutions corresponds to the curve traced out by the one end of a straight line parallel to the axis of z, when the other end traces out the curve given by the first solution.

In the first method of solution the representation is regarded rather as approximate and discontinuous, while in the second it is exact and continuous.

It should be noticed that $z = a\cos\theta = a\cos\{(\omega' - \omega)t - a\}$ is also a simple harmonic motion; and hence the spiral traced out on the surface is the resultant of three particular simple harmonic motions.

In the second method of looking at the problem, it is not necessary that ω and ω' should be nearly equal.

In the case of the second value of z, $z = a\cos(2\omega t + \theta)$, if $\omega' - \omega$ is small, the period for z is just about half of that for x or y.

Seventh Meeting, May 13, 1892.

Professor J. E. A. STEGGALL, M.A., President, in the Chair.

On the Solution of Non-linear Partial Differential Equations of the Second Order.

By JOHN M'COWAN, D.Sc.

§ 1. It is proposed to discuss in this paper partial differential equations involving two independent variables x and y, and a dependent variable z. The method of reduction which is explained can be applied to certain equations involving more than two independent variables, but such application is subject to too many restrictions to be of much general utility.

The usual notation will be employed, viz.:—

$$\frac{\delta z}{\delta x}=p,\quad \frac{\delta z}{\delta y}=q,\quad \frac{\delta^2 z}{\delta x^2}=r,\quad \frac{\delta^2 z}{\delta x\delta y}=s,\quad \frac{\delta^2 z}{\delta y^2}=t.$$

No general method for the solution of partial differential equations of an order higher than the first has yet been obtained. In the case of equations of the second order little towards a general method has been attempted except in the case of the equation, linear in r, s, t,

$$\mathrm{R}r+\mathrm{S}s+\mathrm{T}t=\mathrm{V} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

or the slightly more general form

$$\mathrm{R}r+\mathrm{S}s+\mathrm{T}t+\mathrm{U}(rt-s^2)=\mathrm{V} \quad \ldots \quad \ldots \quad (2$$

in which R, S, T, U, V are functions of x, y, z, p and q.

Methods of more or less generality have been given for the solution of these equations by Monge, Ampere, Laplace, etc.; and these methods comprehend almost all our knowledge of the subject.

It is the principal object of this paper to show how the solution of equations of the second order which are not linear in r, s, and t, may be made to depend on the solution of an equation of the form (1): in many important cases, however, the method will lead to the required solution through equations of the first order only.

§ 2. The general method may be said to consist in the substitution for z of a new dependent variable z', which is a function of x, y, z, p and q; and in the formation of a new equation in x, y, z' and its differential co-efficients p', q', r', s' and t', from the original equation and another of the third order derived from it by differentiation, by the elimination of z, p, q, r, s and t in virtue of the relations between the two sets of differential co-efficients implied in the choice of z'. The equation thus formed will in general be of the second order and linear in r, s, t, but in particular cases it may be of the first order only. From its solution, obtained in any way, the value of z will easily follow by integration.

The nature of the method, however, will be best elucidated by the full consideration of one of the simpler, though remarkably general, cases to which we proceed.

§ 3. Let
$$f(x, y, z, p, q, r, s, t) = 0 \quad \dots \quad \dots \qquad (3)$$
be any non-linear equation of which the solution is required.

Take $z' = q$ for a new dependent variable, write $\delta z'/\delta x = p'$, $\delta^2 z/\delta x^2 = r'$, etc.; then, noting that $s = p'$, $q = t'$, (3) may be written

$$f(x, y, z, p, z', r, p', q') = 0 \quad \dots \quad \dots \quad \dots \qquad (4)$$

Differentiate (4) with respect to y, writing f_x, f_p, f_r etc., for the partial differential co-efficients of f with respect to x, p, r, etc., and we get

$$f_y + z'f_z + p'f_p + q'f_{z'} + r'f_r + s'f_{p'} + t'f_{q'} = 0 \quad \dots \quad \dots \qquad (5)$$

The next step is to eliminate z, p, and r between the equations (4) and (5): it is clear that in general this cannot be done, and thus a limitation is imposed on the form of the equation (3), which so far has been unrestricted. We see at once that for the elimination to be possible, r, p and z must enter into f only as a group $r + Pp + Zz$, where P and Z are functions of x only.

Then the result of the elimination of this group, between (4) and (5) will be of the form

$$Rr' + Ss' + Tt' = V \quad \dots \quad \dots \quad .. \qquad (6)$$

where R, S, T and V are functions of $x . y . z'$, p' and q' only.

If z', that is q, can be obtained by the solution of (6), we have at once $z = \int q dy + X$, where the quantity X, so far an arbitrary function of x, may be easily determined by substitution in the original equation.

The most general form of equation which can be thus solved by taking q as a new dependent variable is as we have seen

$$F(x,\ y,\ q,\ r+Pp+Zz,\ s,\ t)=0 \quad \ldots \quad \ldots \quad (7)$$

where P and Z are functions of x only: but since many equations which are not not of this form may be easily reduced to it by a change of independent variables, the limitation is less than might at first sight appear. Thus, had p been taken as the new dependent variable, we should have obtained for the reducible form

$$F(x,y,p,r,\ s,\ t+Qq+Zz)=0 \quad \ldots \quad \ldots \quad (8)$$

where Q and Z are functions of y only: but this result might be regarded as derived from (7) by an interchange of the independent variables.

§ 4. Another interesting case may be briefly considered.

Take $z'=px+qy-z$ for the new dependent variable.

Hence

$$\left.\begin{aligned} &\frac{\delta z'}{\delta x}=p'=rx+sy,\ \frac{\delta z}{\delta y}=q'=sx+ty \\ &\frac{\delta^2 z}{\delta x^2}=r'=x\frac{\delta r}{\delta x}+y\frac{\delta r}{\delta y}+r \\ &\frac{\delta^2 z}{\delta x\delta y}=s'=x\frac{\delta s}{\delta x}+y\frac{\delta s}{\delta y}+s \\ &\frac{\delta^2 z}{\delta y^2}=t'=x\frac{\delta t}{\delta x}+y\frac{\delta t}{\delta y}+t \end{aligned}\right\} \quad \ldots \quad \ldots \quad (9)$$

Suppose now that the solution of an equation of the form

$$s=f(x,\ y,px+qy-z,\ rx+sy,\ sx+ty) \quad \ldots \quad \ldots \quad (10)$$

is required. In terms of the new variable this may be written

$$s=f(x,y,z',\ p',\ q') \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Operate on both sides of this with $x\dfrac{\delta}{\delta x}+y\dfrac{\delta}{\delta y}+1$; then by (9) we get

$$s'=f+xf_x+yf_y+(p'x+q'y)f_{z'}+r'xf_{p'}+t'yf_{q'} \quad \ldots \quad \ldots \quad (12)$$

an equation, linear in r' s' and t', to give z'.

When z' has been obtained by the solution of (12), (11) will give s, from which z may be obtained by direct integration, except as to an arbitrary function of x, and another of y, which will, however, be at once determined by substitution in the equation $px + qy - z = z'$.

If the equation proposed had been of the general form (3), we should have operated on both sides with $x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y}$ and, by means of the equation thus obtained and the relations (9), eliminated the quantities r, s, t and z: if the original equation was adapted for this method of solution, p and q would be found to have been eliminated also. It may be noted that all equations which contain p, q and z, only in the group $px + qy - z$, are reducible by this method.

The more general case where

$$z' = apx + bqy + cz + hp + kq + \phi(x, y) \quad \dots \quad \dots \quad (13)$$

is taken as the new dependent variable is almost equally simple and interesting. It may be noted that it gives

$$\left.\begin{aligned} p' &= (a + c)p + (h + ax)r + (k + by)s + \phi_x \\ q' &= (b + c)q + (h + ax)s + (k + by)t + \phi_y \end{aligned}\right\} \quad \dots \quad (14)$$

§ 5. The most general substitution may now be considered, in which the new dependent variable may be any function x, y, z, p and q.

Take then

$$z' = \phi(x, y, z, p, q) \quad \dots \quad \dots \quad (15)$$

which gives

$$\left.\begin{aligned} p' &= \phi_x + pq_z + r\phi_p + s\phi_q \\ q' &= \phi_y + q\phi_z + s\phi_p + t\phi_q \end{aligned}\right\} \quad \dots \quad \dots \quad (16)$$

and therefore

$$\left.\begin{aligned} r' &= \phi_q\frac{\delta r}{\delta x} + \phi_q\frac{\delta r}{\delta y} + \mathrm{R} \\ s' &= \phi_p\frac{\delta s}{\delta x} + \phi_q\frac{\delta s}{\delta y} + \mathrm{S} \\ t' &= \phi_p\frac{\delta t}{\delta x} + \phi_q\frac{\delta t}{\delta y} + \mathrm{T} \end{aligned}\right\} \quad \dots \quad \dots \quad (17)$$

where R, S, T are functions of x, y, z, p, q, r, s and t, linear in the three last.

Let the equation to be solved be of the general form (3).

Operate on it with $\phi_p \frac{\delta}{\delta x} + \phi_q \frac{\delta}{\delta y}$: then, by means of the resulting equation and equations (15) to (17), z, r, s, and t can be eliminated from the original equation (3). If on performing this elimination the resulting equation is found to be free of p and q also, we shall have obtained for the determination of z' an equation of the type (1). When, by the solution of this equation, z' has been determined, z may be got by the solution of (15), and the substitution of the resulting value of z in the original equation (3), to determine the precise value of the arbitrary function which would appear in the general solution of (15).

§ 6. One important case of the general method deserves special notice. It may be found that the given equation is of such a form that the quantities z, p, q, r, s, t may be eliminated from it by the relations (15) and (16) without proceeding to a differentiation and the use of the relations (17). In this case the equation proposed reduces to the form

$$f(x, y, z', p', q') = 0, \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

an equation of the first order only to determine z', and thence z by the solution of (15), which is of the first order also.

To illustrate the power of this method one example may be taken. Let the equation proposed for solution be

$$x^2r + 2xys + y^2t = px + qy - z + \mathrm{F}(rx + sy, sx + ty) \quad \ldots \quad (19)$$

Take $z' = px + qy - z$, then by (9), (19) may be written

$$p'x + q'y = z' + \mathrm{F}(p', q') \quad \ldots \quad \ldots \quad (20)$$

an equation belonging to a well known type. The general solution of this equation is got at once by eliminating θ between the equations

$$\left.\begin{array}{l} z' = x\theta + y\Theta - \mathrm{F}(\theta, \Theta) \\ 0 = x + y \dfrac{d\Theta}{d\theta} - \dfrac{d}{d\theta}\mathrm{F}(\theta, \Theta) \end{array}\right\} \quad \ldots \quad \ldots \quad (21)$$

where Θ is an arbitrary function of θ. Then, if $z' = \psi(x, y)$ is the result of this elimination, z is given by

$$px + qy - z = \psi(x. y)$$

and therefore

$$z = \sqrt{(x^2+y^2)}f\left(\frac{y}{x}\right) + \sqrt{(x^2+y^2)}\int\frac{\psi(x,\,y)}{x^2+y^2}\,.\,d.\,\sqrt{(x^2+y^2)} \ldots \quad (22)$$

where in the integration the ratio y/x is to be taken as a constant.

§ 7. It is hardly necessary to remark that the practical application of these methods will often be much facilitated by suitably changing the independent variables. It would be superfluous to give examples of the simplification of equations by the application of the ordinary case of transformation, where the variables are changed from x and y to certain other definite functions of x and y; but there is a peculiar case, sometimes called reciprocation, which deserves special notice owing to its intimate connection with the methods of this paper. The method of reciprocation is well known, and calls therefore for but brief description: it is due, I believe, to Legendre, though De Morgan's name is most commonly associated with it. It consists essentially in the change of the independent variables from x and y to p and q, say x_1 and y_1 for clearness, and the simultaneous change of the dependent variable from z to $px+qy-z$, say z_1. It follows at once that

$$\left.\begin{aligned} p_1 &= \frac{\delta z_1}{\delta x_1} = x \\ q_1 &= \frac{\delta z_1}{\delta y_1} = y \\ r_1 &= \frac{\delta^2 z_1}{\delta x_1^2} = t/\mathrm{J} \\ s_1 &= \frac{\delta^2 z_1}{\delta x_1 \delta y_1} = -s/\mathrm{J} \\ t_1 &= \frac{\delta^2 z_1}{\delta y_1^{\,2}} = r/\mathrm{J} \\ \mathrm{J}_1 &= r_1 t_1 - s_1^{\,2} = 1/\mathrm{J} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (23)$$

where $\mathrm{J} = rt - s^2$.

If the given equation, after transformation by (23) can be solved so as to give z_1 in terms of x_1 and y_1, then z can be obtained in terms of x and y, by the elimination of x_1 and y_1 between the equations

$$\left.\begin{aligned} &z_1 = xx_1 + yy_1 - z \\ &\frac{\delta z_1}{\delta x_1} = x, \quad \frac{\delta z_1}{\delta y_1} = y \end{aligned}\right\} \quad \ldots\ldots \quad \ldots \quad \ldots \quad (24).$$

Two instances will suffice to show the use of reciprocation in connection with the method developed in this paper. For the first, consider the general equation, not explicitly containing the independent variables,

$$f(z,\ p,\ q,\ r,\ s,\ t)=0 \qquad \dots \qquad \dots \qquad (25).$$

If this is transformed by means of the relations (23), it becomes

$$f\{p_1x_1+q_1y_1-z_1,\ x_1,\ y_1,\ t_1/\mathrm{J}_1,\ -s_1/\mathrm{J}_1,\ r_1/\mathrm{J}_1\}=0 \qquad \dots \qquad (26),$$

which is an equation of the general type (10), and is therefore reducible by the substitution considered in § 4.

In this instance the efficacy of reciprocation in adapting equations for easy reduction by our method is well shown; the next is designed to show its use after reduction. An equation considered by Legendre gives a simple example, namely

$$r=f(s,\ t) \qquad \dots \qquad \dots \qquad \dots \qquad (27).$$

This may be regarded as a special case of the general type (7): hence proceeding as in § 3, differentiating with respect to y, we get

$$r'=s'f_s+t'f_t \qquad \dots \qquad \dots \qquad \dots \qquad (28)$$

but

$$s=p',\ t=q'$$

therefore (28) may be written

$$r'=s'f_{p'}+t'f_{q'} \qquad \dots \qquad \dots \qquad \dots \qquad (29)$$

which is the reduced equation linear in r', s', and t'.

This equation we might proceed to solve by any of the methods, such as Mouge's, adapted for the treatment of linear equations, but it is clearly specially suited for reciprocation. The new independent variables will be p' and q', that is s and t, and the new dependent variable $p'x+q'y-z'$, that is $sx+ty-q$, say v; therefore by (23), (29) becomes

$$\frac{\delta^2 v}{\delta s^2}f_t-\frac{\delta^2 v}{\delta s\delta t}f_s-\frac{\delta^2 v}{\delta t^2}=0, \qquad \dots \qquad \dots \qquad (30)$$

a result obtained by Legendre by a special method (*Lacroix*, tom. II., p. 631).

It is easy to combine the methods of change of dependent variable and of reciprocation into one transformation of dependent and independent variables, and, in fact, it was this combination that I first considered, in seeking to extend Legendre's mode of solving (27). I am convinced, however, that in practice it is simpler to use the methods successively, as exemplified in the two preceding examples, just as in the ordinary method of changing variables we usually arrive at the ultimate transformation through many intermediate steps.

It need hardly be added that instead of this complete reciprocation, Routh's method of partial reciprocation or modification may be used in many cases with advantage.

The Plane Triangle ABC: Intimoscribed Circles, etc.

By R. E. ANDERSON, M.A.

§ I. *On an infinite series of Triad Circles derived from the inscribed circle. Determination of a direct relation between* r *and the three radii of the* n^{th} *triad.*

Each of the first triad touches two sides of ABC and the inscribed circle: generally, each circle of the m^{th} triad touches two sides of ABC and also touches one of the circles of the $(m-1)^{th}$ triad.

FIG. 27.

In the diagram only one member of each successive triad is indicated—viz., the circles forming a diminishing series between the in-circle and B.

Let $ON = OY = r$

$$\left.\begin{array}{l} O'N' = r_2' \\ O''N'' = r_2'' \\ O'''N''' = r_3''' \\ \text{etc., etc.} \end{array}\right\} \therefore \left\{\begin{array}{llll} r_2' & r_3' & r_1' & \text{are radii of first triad} \\ r_2' & r_3'' & r_1'' & \text{...... second} \\ r_2''' & r_3''' & r_1''' & \text{...... third} \\ & \text{etc.} & & \text{...... etc.} \end{array}\right.$$

$$\text{and } \rho_2 \quad \rho_3 \quad \rho_1 \quad \text{......} \quad n^{th} \text{ triad,}$$

where in every case the suffix 2 has reference to B, 3 to C, and 1 to A—a rule which also holds for the subsequent sections.

Now $$ON = O'N' + OO'\sin\tfrac{1}{2}B,$$

$$\therefore \quad r = r_2' + (r + r_2')\sin\tfrac{1}{2}B$$

$$r_2' = r_2'' + (r_2' + r_2'')\sin\tfrac{1}{2}B$$

etc., for ever

and
$$\frac{r_2'}{r}=\frac{1-\sin\frac{1}{2}B}{1+\sin\frac{1}{2}B}=\tan^2\tfrac{1}{4}(\pi-B)$$
$$=t_2^2 \text{ suppose.}$$

Thus
$$t_2=\sqrt{\frac{r_2'}{r}}=\sqrt{\frac{r_2''}{r_2'}}=\sqrt{\frac{r_2'''}{r_2''}}=\text{etc.},$$

or
$$\left.\begin{aligned} t_2&=\sqrt[2]{\frac{r_2'}{r}}=\sqrt[4]{\frac{r_2''}{r}}=\sqrt[6]{\frac{r_2'''}{r}}=\text{etc.}\\ \therefore\quad t_3&=\sqrt[2]{\frac{r_3'}{r}}=\sqrt[4]{\frac{r_3''}{r}}=\sqrt[6]{\frac{r_3'''}{r}}=\text{etc.}\\ t_1&=\sqrt[2]{\frac{r_1'}{r}}=\sqrt[4]{\frac{r_1''}{r}}=\sqrt[6]{\frac{r_1'''}{r}}=\text{etc.}\end{aligned}\right\}\begin{aligned}&\text{where}\\ &t_2=\tan\tfrac{1}{4}(\pi-B)\\ &t_3=\tan\tfrac{1}{4}(\pi-C)\\ \text{and }&t_1=\tan\tfrac{1}{4}(\pi-A)\\ \text{or }&\quad\tan\tfrac{1}{4}(A-\pi).\end{aligned}$$

But
$$\tfrac{1}{4}(\pi-B)+\tfrac{1}{4}(\pi-C)+\tfrac{1}{4}(\pi-A)=\frac{\pi}{2}$$

and $\therefore$
$$t_2t_3+t_3t_1+t_1t_2=1.$$

Thus the angular functions are easily eliminated between those three sets of equations so as to establish a direct relation between r and the radii of any triad whatever. Take, for example, the third triad.

$$t_2=\sqrt[6]{\frac{r_2'''}{r}}\quad t_3=\sqrt[6]{\frac{r_3'''}{r}}\quad t_1=\sqrt[6]{\frac{r_1'''}{r}}$$

$$\therefore\qquad t_2t_3+t_3t_1+t_1t_2=1=\sqrt[6]{\frac{r_2'''r_3'''}{r^2}}+\sqrt[6]{\frac{r_3'''r_1'''}{r^2}}+\sqrt[6]{\frac{r_1'''r_2'''}{r^2}}$$

Otherwise
$$\sqrt[3]{r}=\sqrt[6]{r_2'''r_3'''}+\sqrt[6]{r_3'''r_1'''}+\sqrt[6]{r_1'''r'''}$$

Hence these results :—

First triad, $\quad r=\sqrt[2]{r_2'\,r_3'}+\sqrt[2]{r_3'\,r_1'}+\sqrt[2]{r_1'\,r_1'}$

Second ,, $\quad\sqrt[2]{r}=\sqrt[4]{r_2''\,r_3''}+\sqrt[4]{r_3''\,r_1''}+\sqrt[4]{r_1''\,r_2''}$

Third, ,, $\quad\sqrt[3]{r}=\sqrt[6]{r_2'''r_3'''}+\sqrt[6]{r_3'''r_1'''}+\sqrt[6]{r_1'''r_2'''}$

etc. $\qquad$ etc.

n^{th} triad, $\quad\sqrt[n]{r}=\sqrt[2n]{\rho_2\rho_3}+\sqrt[2n]{\rho_3\rho_1}+\sqrt[2n]{\rho_1\rho_2}$

§ II. *On the series of Triad Circles similarly derived from each of the exscribed circles. Determination of the equation between* r$_1$ *(or* r$_2$, *etc.) and the three radii of the* n$^{\text{th}}$ *triad.*

FIG. 28.

Let O and Q_1 in the diagram be centres of the inscribed and exscribed circles, then first triad will touch the ex-circle at X_1, X_2

and X_3. As before, one member only of the successive triads is shown, viz., the series between X_2 and B.

Let $Q_1M = Q_1X_2 = r_1$

$$\left.\begin{array}{r} Q'\,M' = a_2' \\ Q''\,M'' = a_2'' \\ Q'''M''' = a_2''' \\ \text{etc.} \end{array}\right\} \therefore \left\{\begin{array}{llll} a_1' & a_2' & a_3' & \text{are radii of first triad} \\ a_1'' & a_2'' & a_3'' & \ldots\ldots \quad \text{second} \\ a_1''' & a_2''' & a_3''' & \ldots\ldots \quad \text{third} \\ & \text{etc.} & & \qquad\quad\ \text{etc.} \end{array}\right.$$

$$\text{and} \quad a_1 \ a_2 \ a_3 \quad \ldots\ldots \quad n^{th} \text{ triad.}$$

Now $$Q_1M = Q'M' + (Q_1M + Q'M')\sin\tfrac{1}{2}(\pi - B)$$

$$\therefore \quad r_1 = a_2' + (r_1 + a_2')\cos\tfrac{1}{2}B$$

$$\therefore \quad \frac{a_2'}{r_1} = \frac{1 - \cos\frac{1}{2}B}{1 + \cos\frac{1}{2}B} = \frac{\sin^2\frac{1}{4}B}{\cos^2\frac{1}{4}B} = \tan^2\tfrac{1}{4}B$$

$$= t_2^2 \text{ suppose.}$$

Thus $$t_2 = \sqrt{\frac{a_2'}{r_1}} = \sqrt{\frac{a_2''}{a_2'}} = \sqrt{\frac{a_2'''}{a_2''}} = \text{etc.}$$

or
$$\left.\begin{array}{l} t_2 = \sqrt[2]{\dfrac{a_2'}{r_1}} = \sqrt[4]{\dfrac{a_2''}{r_1}} = \sqrt[6]{\dfrac{a_2'''}{r_1}} = \text{etc.} \\ \therefore \quad t_3 = \sqrt[2]{\dfrac{a_3'}{r_1}} = \sqrt[4]{\dfrac{a_3''}{r_1}} = \sqrt[6]{\dfrac{a_3'''}{r_1}} = \text{etc.} \\ t_1 = \sqrt[2]{\dfrac{a_1'}{r_1}} = \sqrt[4]{\dfrac{a_1''}{r_1}} = \sqrt[6]{\dfrac{a_1'''}{r_1}} = \text{etc.} \end{array}\right\} \begin{array}{l} \text{where} \\ \quad t_2 = \tan\frac{1}{4}B \\ \quad t_3 = \tan\frac{1}{4}C \\ \text{and } t_1 = \tan\frac{1}{4}(\pi - A) \\ \text{or} \quad\ \ \tan\frac{1}{4}(A - \pi) \end{array}$$

But $\frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}(A - \pi) = 0$, and the expedient applied in (I.) here fails. I therefore use the theorem that, when the sum of three angles vanishes, the sum of their tangents = the product of their tangents; *i.e.*, $t_1 + t_2 + t_3 = t_1t_2t_3$

$$\therefore \quad \sqrt{\frac{a_1'}{r_1}} + \sqrt{\frac{a_2'}{r_1}} + \sqrt{\frac{a_3'}{r_1}} = \sqrt{\frac{a_1'a_2'a_3'}{r_1^3}} = \frac{1}{r_1}\sqrt{\frac{a_1'a_2'a_3'}{r_1}}$$

Thus

First triad, $$\frac{1}{r_1} = \frac{1}{\sqrt{a_2'a_3'}} + \frac{1}{\sqrt{a_3'a_1'}} + \frac{1}{\sqrt{a_1'a_2'}} = \frac{\sqrt{a_1'} + \sqrt{a_2'} + \sqrt{a_3'}}{\sqrt{a_1'a_2'a_3'}}$$

Second triad, $$\frac{1}{\sqrt[2]{r_1}} = \frac{1}{\sqrt[4]{a_2''a_3''}} + \frac{1}{\sqrt[4]{a_3''a_1''}} + \frac{1}{\sqrt[4]{a_1''a_2''}}$$

Third triad, $$\frac{1}{\sqrt[3]{r_1}} = \frac{1}{\sqrt[6]{a_2'''a_3'''}} + \frac{1}{\sqrt[6]{a_3'''a_1'''}} + \frac{1}{\sqrt[6]{a_1'''a_2'''}}$$

etc. etc.

n^{th} triad, $\dfrac{1}{\sqrt[n]{r_1}} = \dfrac{1}{\sqrt[2n]{a_2a_3}} + \dfrac{1}{\sqrt[2n]{a_3a_1}} + \dfrac{1}{\sqrt[2n]{a_1a_2}}$

For the other two exscribed circles we have of course

$$\frac{1}{\sqrt[n]{r_2}} = \frac{1}{\sqrt[2n]{\beta_2\beta_3}} + \frac{1}{\sqrt[2n]{\beta_3\beta_1}} + \frac{1}{\sqrt[2n]{\beta_1\beta_2}}$$

$$\frac{1}{\sqrt[n]{r_3}} = \frac{1}{\sqrt[2n]{\gamma_2\gamma_3}} + \frac{1}{\sqrt[2n]{\gamma_3\gamma_1}} + \frac{1}{\sqrt[2]{\gamma_1\gamma_2}}$$

$\beta_1\beta_2\beta_3$ being the radii of the n^{th} triad derived from r_2 and
$\gamma_1\gamma_2\gamma_3$ ... ... r_3.

Definition.—Each triad of the four infinite series now discussed is an instance of *intimoscribed circles* (v. infra, Section VI.)

§ III. *On establishing a direct relation between the radii of the twelve circles which form the* nth *triads of the preceding four intimoscribed series.*

From $rs = r_1s_1 = r_2s_2 = r_3s_3 = \Delta$, we derive

$$\sqrt[n]{r}\left(\frac{1}{\sqrt[n]{r_1}} + \frac{1}{\sqrt[n]{r_2}} + \frac{1}{\sqrt[n]{r_3}}\right) = \sqrt[n]{\frac{s_1}{s}} + \sqrt[n]{\frac{s_2}{s}} + \sqrt[n]{\frac{s_3}{s}}, \quad \text{or}$$

$$\frac{1}{\sqrt[n]{r_1}} + \frac{1}{\sqrt[n]{r_2}} + \frac{1}{\sqrt[n]{r_3}} = \frac{\sqrt[n]{\frac{s_1}{s}} + \sqrt[n]{\frac{s_2}{s}} + \sqrt[n]{\frac{s_3}{s}}}{\sqrt[n]{r}}$$

whence substituting from the four general results obtained in (I.) (II.) we obtain

$$\left.\begin{array}{l} \dfrac{1}{\sqrt[n]{a_2a_3}} + \dfrac{1}{\sqrt[2n]{a_3a_1}} + \dfrac{1}{\sqrt[2n]{a_1a_2}} \\ + \dfrac{1}{\sqrt[2n]{\beta_2\beta_3}} + \dfrac{1}{\sqrt[2n]{\beta_3\beta_1}} + \dfrac{1}{\sqrt[2n]{\beta_1\beta_2}} \\ + \dfrac{1}{\sqrt[2n]{\gamma_2\gamma_3}} + \dfrac{1}{\sqrt[2n]{\gamma_3\gamma_1}} + \dfrac{1}{\sqrt[2n]{\gamma_1\gamma_2}} \end{array}\right\} = \frac{\sqrt[n]{\frac{s_1}{s}} + \sqrt{\frac{s_2}{s}} + \sqrt{\frac{s_3}{s}}}{\sqrt[2n]{\rho_2\rho_3} + \sqrt[2n]{\rho_3\rho_1} + \sqrt[2n]{\rho_1\rho_2}}$$

or $$\frac{F(a) + F(\beta) + F(\gamma)}{\sqrt[n]{s_1} + \sqrt[n]{s_2} + \sqrt[n]{s_3}} = \frac{s^{-\frac{1}{n}}}{\sqrt[2n]{\rho_2\rho_3} + \sqrt[2n]{\rho_3\rho_1} + \sqrt[2n]{\rho_1\rho}} = \frac{1}{\sqrt[n]{rs}}$$

where $$F(a) = \frac{1}{\sqrt[2n]{a_2a_3}} + \frac{1}{\sqrt[2n]{a_3a_1}} + \frac{1}{\sqrt[2n]{a_1a_2}}, \quad F(\beta) = \text{ etc.}$$

Three corollaries here occur:—

First, when the sum of the reciprocals of the n^{th} roots of $r_1\ r_2\ r_3$ is multiplied by the n^{th} root of r the result involves only a, b, c.

Second, $$\sqrt{ss_1s_2s_3} = \left(\frac{\sqrt[n]{s_1} + \sqrt[n]{s_2} + \sqrt[n]{s_3}}{F(\alpha) + F(\beta) + F(\gamma)}\right)^n = \sqrt{rr_1r_2r_3}$$

In other words, the area of any Δ is now expressed in terms of the nine radii of the n^{th} triads of intimoscribed circles derived from its three exscribed circles.

Third, if B′ = sum of all the circular areas between the in-circle and B,

$$B' = \pi r^2 \frac{t_2^4}{1 - t_2^4} = \pi r^2 \frac{\sin^4 \frac{1}{4}(\pi - B)}{\cos \frac{1}{2}(\pi - B)} = \pi r^2 \frac{(1 - \sin \frac{1}{2} B)^2}{4 \sin \frac{1}{2} B}$$

$$\therefore \qquad 4B' = \pi r^2 \left(\operatorname{cosec} \frac{B}{2} - 2 + \sin \frac{B}{2}\right)$$

Thus

$$\frac{4(A' + B' + C')}{\pi r^2} = \frac{1}{2} \sqrt{\frac{r}{R}} \left(\sqrt{\frac{a}{s_1}} + \sqrt{\frac{b}{s_2}} + \sqrt{\frac{c}{s_3}}\right)$$
$$+ 2 \sqrt{\frac{R}{r}} \left(\sqrt{\frac{s_1}{a}} + \sqrt{\frac{s_2}{b}} + \sqrt{\frac{s_3}{c}}\right) - 6,$$

$$\therefore \qquad \frac{A' + B' + C'}{\pi r^2} = \frac{1}{8R}(u_1 + u_2 + u_3) + \frac{R}{2}\left(\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3}\right) - \frac{3}{2}$$

where A′ + B′ + C′ = sum of areas of all the intimoscribed circles within ABC, and $u_1\ u_2\ u_3$ are the radii of the superscribed circles (*v.* Section VI.)

Similarly if $A_1' + B_1' + C_1'$, $A_2' + B_2' + C_2'$, $A_3' + B_3' + C_3'$ are respectively the total areas of the intimoscribed triads depending on $r_1\ r_2$ and r_3, we find

$$\frac{A_1' + B_1' + C_1'}{\pi r_1^2} + \frac{A_2' + B_2' + C_2'}{\pi r_2^2} + \frac{A_3' + B_3' + C_3'}{\pi r_3^2} =$$
$$\frac{A' + B' + C'}{\pi r^2} + \frac{1}{4R}(u_1' + u_2' + u_3') + R\left(\frac{1}{u_1'} + \frac{1}{u_2'} + \frac{1}{u_3'}\right) - 3,$$

where $u_1'\ u_2'\ u_3'$ are the radii of the three outer superscribed circles (*v.* Section VI.)

§ IV. *Extension of the principle to any convex* m-*gon which has an inscribed circle* (*radius* r). *Determination of the direct relation* *between* r *and* $(\rho_1\, \rho_2\, \rho_3\, \rho_4 \ldots \rho_m)$ *the radii of the* n*th* *set of intimoscribed circles.*

FIG. 29.

$$\left.\begin{array}{l} OM = r \\ O'M' = r_2' \\ O''M'' = r_2'' \\ \quad \text{etc.} \end{array}\right\}$$

MBN represents an angle of any rectilineal m-gon. If the figure be named ABCDE, etc., then as already shown in (I.)

$$t_2 = \sqrt[2]{\frac{r_2'}{r}} = \sqrt[4]{\frac{r_2''}{r}} = \sqrt[6]{\frac{r_2'''}{r}} = \text{etc.} = \sqrt[2n]{\frac{\rho_2}{r}}$$

$$t_3 = \sqrt[2]{\frac{r_3}{r}} = \sqrt[4]{\frac{r_3'}{r}} = \sqrt[6]{\frac{r_3'''}{r}} = \text{etc.} = \sqrt[2n]{\frac{\rho_3}{r}}$$

$$t_4 = \sqrt[2]{\frac{r_4'}{r}} = \sqrt[4]{\frac{r_4''}{r}} = \text{etc. etc.}$$

where

$$t_2 = \tan\tfrac{1}{4}(\pi - B) = \tan\theta_2$$
$$t_4 = \tan\tfrac{1}{4}(\pi - D) = \tan\theta_4$$
$$t_5 = \tan\tfrac{1}{4}(\pi - E) = \tan\theta_5, \text{ etc. etc.}$$

Now $\theta_1 + \theta_2 + \theta_3 + \ldots \theta_m = \frac{1}{4}$ sum of the "exterior" angles $= \frac{1}{2}\pi$, whatever m may be.

Let, for example, $m = 12$, then

$$\tan(\theta_1 + \theta_2 + \ldots \theta_6) = \cot(\theta_7 + \theta_8 + \ldots \theta_{12})$$

$$\therefore \quad \frac{s_1 - s_3 + s_5}{1 - s_2 + s_4 - s_6} = \frac{1 - s_2' + s_4' - s_6'}{s_1' - s_3' + s_5'}$$

where s_3 = sum of products of the tans of every three of the first six angles, and s_3' = sum of products of tans of every three of the last six angles.

Multiplying up and compressing we get

$$\sigma_2 + \sigma_6 + \sigma_{10} = 1 + \sigma_4 + \sigma_8 + \sigma_{12};$$

where σ_3 = sum of products of tans of every three of the twelve angles. $m = 13$ gives the same result.

Generally whether $m = 2u$ or $2u + 1$

$$\sigma_2 + \sigma_6 + \sigma_{10} + \text{etc.} = 1 + \sigma_4 + \sigma_8 + \text{etc.}$$

or

$$1 - \sigma_2 + \sigma_4 - \sigma_6 + \sigma_8 - \text{etc.} = 0,$$

there being always $u + 1$ terms on left side.

By substituting in that result the values of $t_1\, t_2\, t_3 \ldots t_m$ already found, we can at once reach an equation between r and the m radii of any proposed set of the intimoscribed circles.

Thus for the decagon or hendecagon, n^{th} set,

$$\sigma_2 + \sigma_6 + \sigma_{10} = 1 + \sigma_4 + \sigma_8 \qquad (k)$$

$$\sigma_2 = \sqrt[2n]{\frac{\rho_1\rho_2}{r^2}} + \sqrt[2n]{\frac{\rho_2\rho_3}{r^2}} + \ldots = \frac{\Sigma(\sqrt[2n]{\rho_1\rho_2})}{\sqrt[n]{r}} = \frac{\Sigma_2'}{\sqrt[n]{r}}$$

$$\sigma_4 = \sqrt[2n]{\frac{\rho_1\rho_2\rho_3\rho_4}{r^4}} + \sqrt[2n]{\frac{\rho_1\rho_2\rho_3\rho_5}{r^4}} + \ldots = \frac{\Sigma(\sqrt[2n]{\rho_1\rho_2\rho_3\rho_4})}{\sqrt[n]{r^2}} = \frac{\Sigma_4'}{\sqrt[n]{r^2}}$$

$$\sigma_6 = \frac{\Sigma_6'}{\sqrt[n]{r^3}} \qquad \sigma_8 = \frac{\Sigma_8'}{\sqrt[n]{r^4}}\text{, and } \sigma_{10} = \frac{\Sigma'_{10}}{\sqrt[n]{r^5}}$$

where

$$\Sigma_2' = \sqrt[2n]{\rho_1\rho_2} + \sqrt[2n]{\rho_1\rho_3} + \ldots \text{etc.}$$

$$\Sigma_4' = \sqrt[2n]{\rho_1\rho_2\rho_3\rho_4} + \sqrt[2n]{\rho_1\rho_2\rho_3\rho_5} + \ldots \text{etc.}$$

Substituting in (k)

$$\frac{\Sigma_2'}{\sqrt[n]{r}} + \frac{\Sigma_6'}{\sqrt[n]{r^3}} + \frac{\Sigma_{10}'}{\sqrt[n]{r^5}} = 1 + \frac{\Sigma_4'}{\sqrt[n]{r^2}} + \frac{\Sigma_8'}{\sqrt[n]{r^4}}$$

$$\therefore \qquad 1 - \Sigma_2'p + \Sigma_4'p^2 - \Sigma_6'p^3 + \Sigma_8'p^4 - \Sigma_{10}'p^5 = 0$$

if

$$p = \frac{1}{\sqrt[n]{r}} \quad \text{or } r = \frac{1}{p^n}.$$

The proof just given for the n^{th} set when $m = 10$ or 11 will evidently apply to the most general case, m being any positive integer.

Thus for any m-gon, whether $m = 2u$ or $2u + 1$ we obtain

$$1 - \Sigma_2'p + \Sigma_4'p^2 - \Sigma_6'p^3 + \ldots + (-1)^u\Sigma_{2u}'p^u = 0,$$

an implicit function of r in terms of the m radii of the n^{th} set of intimoscribed circles.

§ V. *On the intimoscribed triads which are derived from the interscribed* circles of any triangle. Determination of an independent equation between the radii of any triad.*

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* *v.* definition *infra*, Section VI.

FIG. 30.

$$\left.\begin{array}{l} RS = n_2 \\ R'S' = n_2' \\ R''S'' = n_2'' \\ \quad \text{etc.} \end{array}\right\}$$

In this system r does not enter but n_2' is derived from n_2, n_2'' from n_2', n_2''' from n_2'', etc., according to the law noted in Section I.

$WZ = n_3 \quad KH = n_1$.

$$\therefore \quad t_2 = \sqrt[2]{\frac{n_2'}{n_2}} = \sqrt[4]{\frac{n_2''}{n_2}} = \sqrt[6]{\frac{n_2'''}{n_2}} = \text{etc.}$$

$$t_3 = \sqrt[2]{\frac{n_3'}{n_3}} = \sqrt[4]{\frac{n''}{n_3}} = \text{etc.}$$

$$t_1 = \sqrt[2]{\frac{n_1'}{n_1}} = \sqrt[4]{\frac{n_1''}{n_1}} = \text{etc.}$$

where $$t_2t_3 + t_3t_1 + t_1t_2 = 1 \qquad (k')$$

Now $$h_1^2n_1 = h_2^2n_2 = h_3^2n_3 = \frac{h_1h_2h_3}{2}, \text{ a constant,}$$

where $$h_1 = 1 + \tan\tfrac{1}{4}A \quad h_2 = 1 + \tan\tfrac{1}{4}B \quad h_3 = 1 + \tan\frac{C}{4}$$

and thus

$$h_1 = 2\sqrt{n_2n_3} \quad h_2 = 2\sqrt{n_3n_1} \quad h_3 = 2\sqrt{n_1n_2}.$$

Substituting in (k')

First triad, $$1 = \sqrt[2]{\frac{n_2'n_3'}{n_2n_3}} + \sqrt[2]{\frac{n_3'n_1'}{n_3n_1}} + \sqrt[2]{\frac{n_1'n_2'}{n_1n_2}}$$

or $$\frac{\sqrt{n_2'n_3'}}{h_1} + \frac{\sqrt{n_3'n_1'}}{h_2} + \frac{\sqrt{n_1'n_2'}}{h_3} = \tfrac{1}{2}$$

Second triad, $$1 = \sqrt[4]{\frac{n_2''n_3''}{n_2n_3}} + \sqrt[4]{\frac{n_3''n_1''}{n_3n_1}} + \sqrt[4]{\frac{n_1''n_2''}{n_1n_2}}$$

or $$\frac{\sqrt[4]{n_2''n_3''}}{\sqrt{h_1}} + \frac{\sqrt[4]{n_3''n_1''}}{\sqrt{h_2}} + \frac{\sqrt[4]{n_1''n_2''}}{\sqrt{h_3}} = \frac{1}{\sqrt{2}}$$

Third triad, $$1 = \sqrt[6]{\frac{n_2'''n_3'''}{n_2n_3}} + \sqrt[6]{\frac{n_3'''n_1'''}{n_3n_1}} + \sqrt[6]{\frac{n_1'''n_2'''}{n_1n_2}}$$

or $$\frac{\sqrt[6]{n_2'''n_3'''}}{\sqrt[3]{h_1}} + \frac{\sqrt[6]{n_3'''n_1'''}}{\sqrt[3]{h_2}} + \frac{\sqrt[6]{n_1'''n_2'''}}{\sqrt[3]{h_3}} = \frac{1}{\sqrt[3]{2}}$$

Generally for m^{th} triad radii $\nu_1\ \nu_2\ \nu_3$.

$$\frac{\sqrt[2m]{\nu_2\nu_3}}{\sqrt[m]{h_1}} + \frac{\sqrt[2m]{\nu_3\nu_1}}{\sqrt[m]{h_2}} + \frac{\sqrt[2m]{\nu_1\nu_2}}{\sqrt[m]{h_3}} = \frac{1}{\sqrt[m]{2}}$$

[*Note.*—$h_1\ h_2\ h_3$ can readily be expressed severally in terms of a, b, c.

Thus $$h_1^2 = (1+\tan\tfrac{1}{4}A)^2 = \sec^2\tfrac{1}{4}A + 2\tan\tfrac{1}{4}A$$
$$= \frac{1+\sin\frac{1}{2}A}{\cos^2\frac{1}{4}A} = \frac{2(1+\sin\frac{1}{2}A)}{1+\cos\frac{1}{2}A} = \frac{2(\sqrt{bc}+\sqrt{s_2s_3})}{\sqrt{bc}+\sqrt{ss_1}}.]$$

§ VI. *On a classification of the circles belonging to* ABC, *with notes of a few of their properties.*

1. – three leading circles, the circumscribed, medioscribed and inscribed.
2. – three exscribed circles.
3. – three interscribed circles.
4. – six superscribed circles.
5. – twelve insuperscribed circles.
6. – twelve introscribed circles.
7. – fifteen intimoscribed circles.
8. – four tercentroscribed circles.
9. – fifteen supermedianscribed circles.
10. – fifteen intermedianscribed circles.

(3.) *Interscribed Circles* (radii $n_1 n_2 n_3$). Each touches two sides of ABC, and also touches the other interscribed circles.

$$h_1^2 n_1 = h_2^2 n_2 = h_3^2 n_3 = \tfrac{1}{2} h_1 h_2 h_3, \text{ a constant,} \qquad \text{i.}$$

where $$h_2^2 = \frac{2(\sqrt{ca}+\sqrt{s_3s_1})}{\sqrt{ca}+\sqrt{ss_2}} \quad \text{(See V.)}$$

$$2\{\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3} \pm \sqrt{(n_1+n_2+n_3)}\}^2 = h_1h_2h_3 \qquad \text{ii.}$$

(4.) *Superscribed Circles.* These have for their diameters the lines joining the four centres O Q_1 Q_2 Q_3. Taking the inner three (radii $u_1\ u_2\ u_3$) we have

$$u_1u_2u_3 = 2rR^2 = \frac{1}{2r}\left(\frac{abc}{a+b+c}\right)^2 \qquad \text{iii.}$$

$$v_1^2 + u_2^2 + u_3^2 = \frac{R}{s}(ar_1 + br_2 + cr_3) \qquad \text{iv.}$$

$$\frac{a}{u_1^2} + \frac{b}{u_2^2} + \frac{c}{u_3^2} = \frac{(a+b+c)^2}{abc} \qquad \text{v.}$$

$$\frac{u_2^2 u_3^2}{bc} + \frac{u_3^2 u_1^2}{ca} + \frac{u_1^2 u_2^2}{ab} = R^2 \qquad \text{vi.}$$

(5.) *Insuperscribed Circles.* These have for their diameters the lines joining the angular points of ABC with the four centres. Taking the two inner groups, diameters AO, BO, CO (radii v_1' v_2' v_3') and AQ_1 BQ_2 CQ_3 (radii V_1' V_2' V_3') we have

$$\frac{4v_1'v_2'v_3'}{r} = \frac{abc}{a+b+c} = \frac{u_1u_2u_3}{R} \qquad \text{vii.}$$

$$\left.\begin{aligned} 4(av_1'^2 + bv_2'^2 + cv_3'^2) &= abc \\ \text{or} \quad \frac{v_1'^2}{bc} + \frac{v_2'^2}{ca} + \frac{v_3}{ab} &= \tfrac{1}{4} \end{aligned}\right\} \qquad \text{viii.}$$

$$\frac{aV_1'^2 + bV_2'^2 + cV_3'^2}{R} = \frac{3abc - a^3 - b^3 - c^3}{8r} \qquad \text{ix.}$$

$$\frac{bc}{V_1'^2} + \frac{ca}{V_2'^2} + \frac{ab}{V_3'^2} = 4,$$

$$\text{or} \quad \frac{1}{aV_1'^2} + \frac{1}{bV_2'^2} + \frac{1}{cV_3'^2} = \frac{4}{abc} \qquad \text{x.}$$

(6.) *Introscribed Circles.* These twelve are respectively inscribed to the quadrilaterals which the circles of (5.) circumscribe. Taking the groups (radii v_1 v_2 v_3 and V_1 V_2 V_3) corresponding to those selected in the preceding paragraph, we have

$$\frac{v_1}{r - v_1} + \frac{v_2}{r - v_2} + \frac{v_3}{r - v_3} = \frac{s}{r} \qquad \text{xi.}$$

$$\frac{1}{V_1} + \frac{1}{V_2} + \frac{1}{V_3} = \frac{3}{s} + \frac{1}{r} \qquad \text{xii.}$$

(7.) *Intimoscribed Circles.* Five groups, each circle touching two sides of ABC and also touching either the inscribed circle or one of the exscribed circles or one of the interscribed circles. Each group is the first triad of an infinite series, all the circles of which may be termed "intimoscribed." See Sections I., II., III., and V. for their properties.

(8.) *Tercentroscribed Circles.* Each passes through three of the four centres O Q_1 Q_2 Q_3; and has radius $R' = 2R$; the triangle ABC being orthic to $Q_1Q_2Q_3 (= \Delta')$. Let a', b', c', be the sides of Δ'.

$$\Delta' = \frac{abc}{2r} = R's = \frac{2R}{r}\Delta \qquad \text{xiii.}$$

$$(a'^2bc + b'^2ca + c'^2ab)^{\frac{1}{2}} = \frac{abc}{r} = R'(a+b+c) \qquad \text{xiv.}$$

$$\frac{a'^2 + b'^2 + c'^2}{bc + ca + ab} = \frac{2R'}{r} \qquad \text{xv.}$$

$$a'^2 + b'^2 + c'^2 + 4(u_1^2 + u_2^2 + u_3^2) = 12R'^2 \qquad \text{xvi.}$$

(9, 10.) These two groups are the circles circumscribed about and inscribed in the 15 triangles formed by the medians (m_1 m_2 m_3). Let the in-radii of the smaller triangles round G be as marked; those of GBC, GCA, GAB be r_4 r_5 r_6 respectively; and those of AM_1B, AM_1C, BM_2C, BM_2A, CM_3A, CM_3B be r_4', r_4'', r_5', r_5'', r_6', r_6'' respectively—the circum-radius in each case being known by changing r to R.

Fig. 31.

$$\frac{1}{r_1'} + \frac{1}{r_2'} + \frac{1}{r_3'} = \frac{1}{r_1''} + \frac{1}{r_2''} + \frac{1}{r_3''} = \frac{3}{2}\left(\frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6} - \frac{1}{r}\right)$$

$$= \frac{3}{r} + \frac{3(m_1 + m_2 + m_3)}{\Delta} \qquad \text{xvii.}$$

$$R_1'R_2'R_3' = R_1''R_2''R_3'' = \frac{R_4R_5R_6}{128}\left(\frac{abc}{R}\right)^2 = \frac{R}{54}\, m_1^2 m_2^2 m_3^2 \qquad \text{xviii.}$$

$$\frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6} = \frac{3}{r} + \frac{2(m_1 + m_2 + m_3)}{\Delta} \qquad \text{xix.}$$

$$R_4R_5R_6 = 64\left(\frac{R}{3}\right)^3\left(\frac{m_1m_2m_3}{abc}\right)^2 \qquad \text{xx.}$$

$$\frac{1}{r_4'} + \frac{1}{r_5'} + \frac{1}{r_6'} = \frac{1}{r_4''} + \frac{1}{r_5''} + \frac{1}{r_6''}$$

$$= \frac{3}{r} + \frac{m_1 + m_2 + m_3}{\Delta} = \tfrac{1}{2}\left(\frac{3}{r} + \frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6}\right) \qquad \text{xxi.}$$

$$R_4'R_5'R_6' = R_4''R_5''R_6'' = \left(\frac{R}{2}\right)^2 \frac{m_1 m_2 m_3}{\Delta} \qquad \text{xxii.}$$

$$\frac{R_4'R_5'R_6'}{R_1'R_2'R_3'} = \frac{54\,R^5}{a^2b^2c^2} \qquad \text{xxiii.}$$

$$\therefore \quad \frac{R_4'R_4''}{R_4} \cdot \frac{R_5'R_5''}{R_5} \cdot \frac{R_6'R_6''}{R_6} = \left(\frac{3R}{4}\right)^3,$$

or

$$\frac{RR_4}{R_4'R_4''} \cdot \frac{RR_5}{R_5'R_5''} \cdot \frac{RR_6}{R_6'R_6''} = \frac{64}{27} \qquad \text{xxiv.}$$

§ VII. *On some algebraic results which symmetrically involve* a, b, c *and* l_1, l_2, l_3 (*the bisectors of* A, B, C *drawn to the opposite sides*).

(1),
$$\frac{l_1^2}{ss_1} = \frac{4bc}{(b+c)^2} \quad \therefore \quad \frac{l_1^2}{bc} = \frac{(b+c+a)(b+c-a)}{(b+c)^2} = 1 - \frac{a^2}{(b+c)^2}$$

Thus
$$1 = \frac{l_1^2}{bc} + \frac{a^2}{(b+c)^2} = \frac{l_2^2}{ca} + \frac{b^2}{(c+a)^2} = \frac{l_3^2}{ab} + \frac{c^2}{(a+b)^2}\,.^*$$

(2),
$$\left(\frac{1}{b}+\frac{1}{c}\right)\frac{b^2-c^2}{l_2^2l_3^2} + \left(\frac{1}{c}+\frac{1}{a}\right)\frac{c^2-a^2}{l_3^2l_1^2} + \left(\frac{1}{a}+\frac{1}{b}\right)\frac{a^2-b^2}{l_1^2l_2^2} = 0.$$

(3),
$$\frac{\frac{1}{b^2}-\frac{1}{c^2}}{(s-b)(s-c)}\left(\frac{b+c}{l_2^2l_3^2}\right) + \frac{\frac{1}{c^2}-\frac{1}{a^2}}{(s-c)(s-a)}\left(\frac{c+a}{l_3^2l_1^2}\right) + \frac{\frac{1}{a^2}-\frac{1}{b^2}}{(s-a)(s-b)}\left(\frac{a+b}{l_1^2l_2^2}\right) = 0.$$

If $2\sigma^2 = a^2 + b^2 + c^2$,

(4),
$$a^2(\sigma^2-a^2)^2 + b^2(\sigma^2-b^2)^2 + c^2(\sigma^2-c^2)^2 + 2(\sigma^2-a^2)(\sigma^2-b^2)(\sigma^2-c^2) = a^2b^2c^2, \text{ and}$$

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[*] These values of l_2 and l_3 give a short *direct* proof that the Δ is isosceles when $l_2 = l_3$. Thus $ca\left(1-\frac{b^2}{(c+a)^2}\right) = ab\left(1-\frac{c^2}{(a+b)^2}\right)$

$$\therefore \quad c(a+b)^2(s-b) = b(c+a)^2(s-c)$$

$$\therefore \quad (b-c)(abc+bcs+a^2s) = 0$$

$$\therefore \quad b-c=0, \text{ or } b=c.$$

(5),
$$\frac{R}{4P_1P_2P_3}=\frac{1}{\sigma^2-a^2}+\frac{1}{\sigma^2-b^2}+\frac{1}{\sigma^2-c^2}-\frac{4\Delta^2}{(\sigma^2-a^2)(\sigma^2-b^2)(\sigma^2-c^2)}$$

and
$$\left(\frac{\sigma^2}{a^2}-1\right)\left(\frac{\sigma^2}{b^2}-1\right)\left(\frac{\sigma^2}{c^2}-1\right)=\frac{P_u}{R}\cdot\frac{P_2}{R}\cdot\frac{P_3}{R},$$

$P_1P_2P_3$ being the perpendiculars from the circum-centre.

(6),
$$1=\frac{1}{\left(1-\frac{a}{b}\right)\left(1-\frac{a}{c}\right)}+\frac{1}{\left(1-\frac{b}{c}\right)\left(1-\frac{b}{a}\right)}+\frac{1}{\left(1-\frac{c}{a}\right)\left(1-\frac{c}{b}\right)}$$
$$=\frac{1}{\left(\frac{b}{a}-1\right)\left(\frac{c}{a}-1\right)}+\frac{1}{\left(\frac{c}{b}-1\right)\left(\frac{a}{b}-1\right)}+\frac{1}{\left(\frac{a}{c}-1\right)\left(\frac{b}{c}-1\right)}$$

(7),
$$(f+g+h)^{n+1}r^n=f^{n+1}r_1{}^n=g^{n+1}r_2{}^n+h^{n+1}r_3{}^n$$

if only
$$\frac{f}{s_1}=\frac{g}{s_2}=\frac{h}{s_3}$$

(8),
$$\frac{a+x}{a(a-b)(a-c)}+\frac{b+x}{b(b-c)(b-a)}+\frac{c+x}{c(c-u)(c-b)}=\frac{x}{abc},$$

x being a line of any length; when also

(9),
$$\frac{(a-x)^2}{(a-b)(a-c)}+\frac{(b-x)^2}{(b-c)(b-a)}+\frac{(c-x)^2}{(c-a)(c-b)}=1.$$

(10),
$$\frac{bc+ca+ab}{2(a+b+c)}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=1+Rr\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

or
$$(bc+ca+ab)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+4s.$$

(11),
$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=3,\quad\text{when}$$
$$\frac{a+bm}{b+cn}=\frac{b+cm}{c+an}=\frac{c+am}{a+bn}.$$

(12), For any scalene Δ
$$\frac{\frac{1}{a}-\frac{1}{b}}{a^2+b^2}+\frac{\frac{1}{b}-\frac{1}{c}}{b^2+c^2}+\frac{\frac{1}{c}-\frac{1}{a}}{c^2+a^2}=0,$$

if
$$a^3(b-c)+a^2(b^2-c^2)+a(b^3-c^3)+(b^4-c^4)=0.$$

(13), When s has a vanishing value, not otherwise,
$$\frac{a^2}{(b+2c)(c+2b)}+\frac{b^2}{(c+2a)(a+2c)}+\frac{c^2}{(a+2b)(b+2a)}=1.$$

Eighth Meeting, June 10, 1892.

Professor J. E. A. Steggall, M.A., President, in the Chair.

Integrals of the form $$\int_0^{2\pi}\log\left\{(x-a\cos\theta)^2+(y-b\sin\theta)^2\right\}{\cos\atop\sin}m\theta d\theta$$ and allied Integrals.

By George A. Gibson, M.A.

1. If θ be the eccentric angle of a point P on an ellipse whose semi-axes are a and b and if x, y be the rectangular co-ordinates of a point Q, then $PQ^2=(x-a\cos\theta)^2+(y-b\sin\theta)^2$. When $a=b$ and $x^2+y^2=f^2$, $PQ^2=f^2-2af\cos\theta+a^2$, if the line from which θ is measured passes through Q. The integral

$$\int_0^{2\pi}\log(f^2-2af\cos\theta+a^2)\cos m\theta d\theta$$

is a well-known one. I propose in the present paper to consider the more general form which the integral takes when P lies on an ellipse. Some of the results I establish are proved in Roberts's *Integral Calculus*, pp. 199-201, by a totally different method; others seem to me to be new.

I use the following notation throughout.

$$a=c\cosh u \qquad x=\alpha\cos\phi=c\cosh v\cos\phi$$
$$b=c\sinh u \qquad y=\beta\sin\phi=c\sinh v\sin\phi$$

so that $$c^2=a^2-b^2=\alpha^2-\beta^2.$$

P and Q thus lie on two confocal ellipses. Putting $PQ=R$, then R^2 may be thrown into the form

$$R^2=c^2\{\cosh(u+v)-\cos(\theta+\phi)\}\{\cosh(u-v)-\cos(\theta-\phi)\}$$
$$=c^2\Delta \text{ suppose.}$$

This expression for R^2 is given in Greenhill's *Calculus* (2nd ed., p. 372). Before noticing this form I had used complex factors, which are rather more troublesome to work with. By means of the factors the functions to be integrated are readily expressed by a series.

The usual methods of expansion give

$$\log(\mathrm{A}-\cos x)=\log\frac{\mathrm{A}+\sqrt{(\mathrm{A}^2-1)}}{2}-2\sum_1^{\infty}\{\mathrm{A}-\sqrt{(\mathrm{A}^2-1)}\}^n\frac{\cos nx}{n}$$

A being greater than unity, and the root sign being so chosen that $\mathrm{mod}\{\mathrm{A}-\sqrt{(\mathrm{A}^2-1)}\}<1$. For example, if $\mathrm{A}=\cosh(u-v)$ and $u<v$, $\mathrm{A}-\sqrt{(\mathrm{A}^2-1)}=e^{u-v}$, $\mathrm{A}+\sqrt{(\mathrm{A}^2-1)}=e^{v-u}$.

2. $$\int_0^{2\pi}\log(c^2\Delta)d\theta.$$

The integral is equal to

$$\int_0^{2\pi}\log c^2 d\theta+\int_0^{2\pi}\log\{\cosh(u+v)-\cos(\theta+\phi)\}d\theta+\int_0^{2\pi}\log\{\cosh(u-v)-\cos(\theta-\phi)\}d\theta$$

$$=4\pi\log c+\int_0^{2\pi}\log\{\cosh(u+v)-\cos\theta\}d\theta+\int_0^{2\pi}\log\{\cosh(u-v)-\cos\theta\}d\theta$$

since the function to be integrated has the period 2π and ϕ is real.

(i.) Suppose $u>v$, that is, suppose Q inside the ellipse on which P lies.

$$\therefore\quad \int_0^{2\pi}\log(c^2\Delta)d\theta=4\pi\log c+2\pi(u+v-\log 2)+2\pi(u-v-\log 2)$$

$$=4\pi\left(u+\log\frac{c}{2}\right)$$

$$=4\pi\log\frac{a+b}{2}$$

since $u=\log e^u=\log(\cosh u+\sinh u)$. The integrals are of course obtained by expanding the logarithms, noting that $\int_0^{2\pi}\cos n\theta d\theta=0$.

It will be observed that the value is independent of v, that is of the position of Q within the ellipse.

(ii.) Suppose $u < v$, that is, Q outside the ellipse ; then

$$\int_0^{2\pi}\log(c^2\Delta)d\theta = 4\pi\log c + 2\pi(u+v-\log 2) + 2\pi(v-u-\log 2)$$

$$= 4\pi\left(v + \log\frac{c}{2}\right)$$

$$= 4\pi\log\frac{\alpha+\beta}{2}\,.$$

In this case the integral is independent of u.

(iii.) Suppose $u = v$. The integral is still determinate, and we get

$$\int_0^{2\pi}\log(c^2\Delta)d\theta = 4\pi\log\frac{a+b}{2}\,.$$

By supposing the ellipse on which P lies to become a circle of radius a, we get the well-known values

$$\int_0^{2\pi}\log(\mathrm{PQ}^2)d\theta = 4\pi\log a \text{ or } 4\pi\log f$$

according as Q lies inside or outside the circle, f being the distance of Q from the centre.

3. $$\int_0^{2\pi}\log(c^2\Delta)\cos m\theta d\theta$$

where m is a positive integer.

If $x = \theta \pm \phi$, $\cos nx = \cos n\theta\cos n\phi \mp \sin n\theta\sin n\phi$. Hence $\log(c^2\Delta)$ may be expanded in a series involving the terms $\mathrm{C}_n\cos n\theta$ and $\mathrm{S}_n\sin n\theta$. But

$$\int_0^{2\pi}\cos n\theta\cos m\theta d\theta = 0 \text{ if } m \neq n, \text{ but} = \pi \text{ if } m = n$$

and $$\int_0^{2\pi}\sin n\theta\cos m\theta d\theta = 0.$$

Therefore expanding the logarithms we get

$$\int_0^{2\pi} \log\{\cosh(u+v) - \cos(\theta+\phi)\}\cos m\theta d\theta$$

$$= -\frac{2\pi}{m}\cos m\phi . e^{-m(u+v)}$$

$$\int_0^{2\pi} \log\{\cosh(u-v) - \cos(\theta-\phi)\}\cos m\theta d\theta$$

$$= -\frac{2\pi}{m}\cos m\phi . e^{-m(u-v)} \text{ if } u>v$$

$$= -\frac{2\pi}{m}\cos m\phi e^{+m(u-v)} \text{ if } u<v.$$

Hence $$\int_0^{2\pi} \log(c^2\Delta)\cos m\theta d\theta$$

$$= -\frac{4\pi}{m}\cos m\phi \cosh mv(\cosh mu - \sinh mu) \text{ if } u>v$$

$$= -\frac{4\pi}{m}\cos m\phi \cosh mu(\cosh mv - \sinh mv) \text{ if } u<v$$

If $u=v$, the integral is still determinate, and has for value

$$-\frac{4\pi}{m}\cos m\phi \cosh mu(\cosh mu - \sinh mu).$$

The values of the integral in this case depend on both the co-ordinates of Q.

In order to deduce the values when $a=b$, it may be noticed that $\cosh mv(\cosh mu - \sinh mu)$

$$= \tfrac{1}{2}\{(\cosh v + \sinh v)^m + (\cosh v - \sinh v)^m\}\{\cosh u - \sinh u\}^m$$

$$= \tfrac{1}{2}\frac{\{(\alpha+\beta)^m + (\alpha-\beta)^m\}(a-b)^m}{c^{2m}}$$

$$= \tfrac{1}{2}\frac{(\alpha+\beta)^m + (\alpha-\beta)^m}{(a+b)^m}$$

Hence if Q be at distance f from the centre of the circle, we get

$$\int_0^{2\pi} \log\{f^2 - 2af\cos(\theta-\phi) + a^2\}\cos m\theta d\theta = -\frac{2\pi}{m}\cos m\phi\frac{f^m}{a^m}$$

$$\text{or} \quad -\frac{2\pi}{m}\cos m\phi\frac{a^m}{f^m}$$

according as f is less or greater than a, and is equal to either value when f and a are equal.

4.
$$\int_0^{2\pi}\log(c^2\Delta)\sin m\theta d\theta,$$

where m is a positive integer.

As in the preceding example, it may be shown that

$$\begin{aligned}\int_0^{2\pi}\log(c^2\Delta)\sin m\theta d\theta &= -\frac{4\pi}{m}\sin m\phi\sinh mv(\cosh mu - \sinh mu) \text{ if } u>v\\ &= -\frac{4\pi}{m}\sin m\phi\sinh mu(\cosh mv - \sinh mv) \text{ if } u<v\\ &= -\frac{4\pi}{m}\sin m\phi\sinh mu(\cosh mu - \sinh mu) \text{ if } u=v\end{aligned}$$

When $a=b$, the values are as in the corresponding cases of § 3 with $\sin m\phi$ instead of $\cos m\phi$.

5.
$$\int_0^{\pi}\log(c^2\Delta)\genfrac{}{}{0pt}{}{\sin}{\cos}m\theta d\theta$$

where m is a positive integer.

In general these integrals have to be expressed in series; the particular case

$$\int_0^{\pi}\log\{(x-a\cos\theta)^2+b^2\sin^2\theta\}\cos m\theta . d\theta \ (i.e., \ \phi=0)$$

is, however, simply half the integral of § 3 with ϕ put equal to zero. As the series can be readily expressed, it is needless to write them down.

6. By differentiation of the integrals, others may be deduced. Thus

$$\begin{aligned}&\int_0^{\pi}\log\{(x-a\cos\theta)^2+b^2\sin^2\theta\}\cos m\theta d\theta\\ &\qquad = -\frac{2\pi}{m}\cosh mv(\cosh mu - \sinh mu) \quad u>v\\ &\qquad = -\frac{\pi}{m}\cdot\frac{\{x+\sqrt{(x^2-a^2+b^2)}\}^m+\{x-\sqrt{(x^2-a^2+b^2)}\}^m}{(a+b)^m}\end{aligned}$$

since in this case $\alpha=x$, $\beta=\sqrt{\alpha^2-c^2}=\sqrt{(x^2-a^2+b^2)}$.

Now differentiate with respect to b.

$$\because \int_0^\pi \frac{\sin^2\theta\cos m\theta d\theta}{(x-a\cos\theta)^2+b^2\sin^2\theta}$$

$$=\frac{\pi}{2}\cdot\frac{\{x+\sqrt{(x^2-a^2+b^2)}\}^m+\{x-\sqrt{(x^2-a^2+b^2)}\}^m}{b(a+b)^{m+1}}$$

$$-\frac{\pi}{2}\cdot\frac{\{x+\sqrt{(x^2-a^2+b^2)}\}^{m-1}-\{x-\sqrt{(x^2-a^2+b^2)}\}^{m-1}}{(a+b)^m.\sqrt{(x^2-a^2+b^2)}}$$

If $a=b$, this becomes

$$\int_0^\pi \frac{\sin^2\theta\cos m\theta d\theta}{x^2-2ax\cos\theta+a^2}=-\frac{\pi}{4}(a^2-x^2)\frac{x^{m-2}}{a^{m+2}} \qquad x<a$$

If $x>a$, we get

$$\int_0^\pi \frac{\sin^2\theta\cos m\theta d\theta}{(x-a\cos\theta)^2+b^2\sin^2\theta}=\frac{\pi}{2}\,\frac{(a+b)^m+(a-b)^m}{\sqrt{(x^2-a^2+b^2)}\{x+\sqrt{(x^2-a^2+b^2)}\}^{m+1}}$$

$$-\frac{\pi}{2}\,\frac{(a+b)^{m-1}-(a-b)^{m-1}}{b\{x+\sqrt{(x^2-a^2+b^2)}\}^m}$$

and if $a=b$

$$\int_0^\pi \frac{\sin^2\theta\cos m\theta d\theta}{x^2-2ax\cos\theta+a^2}=-\frac{\pi}{4}(x^2-a^2)\frac{a^{m-2}}{x^{m+2}} \qquad x>a.$$

Similarly by differentiating the integrals of §§ 3 and 4 with respect to u or v, various integrals may be obtained.

7. $$\int_0^{2\pi}\frac{d\theta}{c^2\Delta.}$$

In this case $u \mp v$. We may write

$$\cosh(u+v)-\cos(\theta+\phi)=-\frac{\cos\phi+i\sin\phi}{2x}(x-x_1)(x-x_2)$$

where $x=e^{i\theta}$, $\mathrm{mod}\,x_1<1$, $\mathrm{mod}\,x_2>1$, x_1, x_2 being the roots of

$$(\cos\phi+i\sin\phi)x^2-2\cosh(u+v).x+\cos\phi-i\sin\phi=0.$$

Similarly

$$\cosh(u-v)-\cos(\theta-\phi)=-\frac{\cos\phi-i\sin\phi}{2x}(x-x_3)(x-x_4)$$

where $\mathrm{mod}\,x_3<1$, $\mathrm{mod}\,x_4>1$, and x_3, x_4 are roots of

$$(\cos\phi-i\sin\phi)x^2-2\cosh(u-v)x+\cos\phi+i\sin\phi=0.$$

Hence
$$\frac{1}{\Delta}=\frac{4x^2}{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}$$
$$=\frac{A}{x-x_1}+\frac{B}{x-x_2}+\frac{C}{x-x_3}+\frac{D}{x-x_4}\,.$$

It is obvious on expanding $A/(x-x_1)$, etc., that

$$\int_0^{2\pi}\frac{d\theta}{x-x_1}=0=\int_0^{2\pi}\frac{d\theta}{x-x_3}\,;\ \int_0^{2\pi}\frac{d\theta}{x-x_2}=-\frac{2\pi}{x_2}\,,\ \int_0^{2\pi}\frac{d\theta}{x-x_4}=-\frac{2\pi}{x_4}\,.$$

But $$B=\frac{4x_2^2}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}\,,\ D=\frac{4x_4^2}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$$\therefore\quad \frac{B}{x_2}+\frac{D}{x_4}=\frac{4(x_1x_3-x_2x_4)}{(x_2-x_1)(x_2-x_3)(x_1-x_4)(x_3-x_4)}$$
$$=-\frac{2\sinh 2u}{(\cosh 2u-\cosh 2v)(\cosh 2u-\cos 2\phi)}\qquad u>v$$
$$=-\frac{2\sinh 2v}{(\cosh 2v-\cosh 2u)(\cosh 2v-\cos 2\phi)}\qquad u<v$$

$$\therefore\quad \int_0^{2\pi}\frac{d\theta}{c^2\Delta}=-\frac{2\pi}{c^2}\left(\frac{B}{x_2}+\frac{D}{x_4}\right)$$
$$=\frac{4\pi\sinh 2u}{c^2(\cosh 2u-\cosh 2v)(\cosh 2u-\cos 2\phi)}\qquad u>v$$
$$=\frac{4\pi\sinh 2v}{c^2(\cosh 2v-\cosh 2u)(\cosh 2v-\cos 2\phi)}\qquad u<v$$

If these values be expressed in terms of x, y, α, β, a, b, we get

$2\pi ab/(a^2b^2-b^2x^2-a^2y^2)$ and $2\pi\alpha\beta/(\alpha^2-a^2)(\alpha^2+\beta^2-x^2-y^2)$
respectively.

8. $$\int_0^{2\pi}\frac{\cos m\theta\, d\theta}{c^2\Delta}$$

where m is a positive integer.

Writing $1/\Delta$ as in § 7, expanding and integrating after multiplication by $\cos m\theta$, we get

$$\int_0^{2\pi}\frac{\cos m\theta\, d\theta}{c^2\Delta}=\frac{\pi}{c^2}\left\{Ax_1^{m-1}-\frac{B}{x_2^{m+1}}+Cx_3^{m-1}-\frac{D}{x_4^{m+1}}\right\}$$

The reduction after substituting for A, B, C, D is somewhat tedious, but we get, if $u > v$, for the value of the integral

$$4\pi e^{-mu} \cdot \frac{\cosh mv \sin m\phi \sin 2\phi(\cosh 2u - \cosh 2v) + \sinh mv \sinh 2v \cos m\phi (\cosh 2u - \cos 2\phi) + \cosh mv \sinh 2u \cos m\phi (\cosh 2v - \cos 2\phi)}{c^2(\cosh 2u - \cosh 2v)(\cosh 2u - \cos 2\phi)(\cosh 2v - \cos 2\phi)}$$

If $u < v$, then u and v must be interchanged.

If we suppose $a = b$, and $a = \beta = \sqrt{x^2 + y^2} = f$ as before, then

$$\int_0^{2\pi} \frac{\cos m\theta d\theta}{f^2 - 2af\cos(\theta - \phi) + a^2} = \frac{2\pi f^m \cos m\phi}{a^m(a^2 - f^2)} \text{ or } \frac{2\pi a^m \cos m\phi}{f^m(f^2 - a^2)}$$

according as $f <$ or $> a$.

If we had taken $\phi = 0$, the integral would have been much more easily evaluated, since

$$\int_0^{2\pi} \cos m\theta . d\theta / (\mathrm{A} - \cos\theta)$$

is very easily evaluated.

Matthew Stewart's Theorem.

By J. S. Mackay, M.A., LL.D.

In 1746, when he was a candidate for the Chair of Mathematics in the University of Edinburgh rendered vacant by the death of Maclaurin, Matthew Stewart published his first work, *Some General Theorems of considerable use in the higher parts of Mathematics.* In the preface to it he states that "the theorems contained in the following sheets . . . . are entirely new, save one or two at most," but he does not specify the two. They are*

Theorem 1.

If in triangle ABC any straight line AD be drawn to BC, and DE, DF be drawn parallel to AC, AB, and meeting AB, AC in E, F, then

$$AD^2 + BD \cdot CD = AB \cdot AE + AC \cdot AF.$$

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* The enunciations of these theorems have been modernised.

Theorem 2.

If in triangle ABC any straight line AD be drawn to BC, then

$$AB^2 \cdot CD + AC^2 \cdot BD = BC \cdot BD \cdot CD + AD^2 \cdot BC.$$

It is the second of these that it is usual to call Stewart's theorem, and Stewart employs the first one to prove it. There is good reason, however, for believing that the theorem, though it was first published by Stewart, was discovered by Robert Simson. In the second book of Simson's *Loci Plani*, which was published in 1749, the tenth lemma (p. 156) is to the following effect.*

If ABC be a triangle, and from the vertex to the base BC there be drawn any straight line AD, and DE be any straight line, then

$$\frac{AB^2 \cdot CD}{DE} + \frac{AC^2 \cdot BD}{DE} = \frac{BD^2 \cdot CD}{DE} + \frac{CD^2 \cdot BD}{DE} + \frac{AD^2 \cdot BC}{DE}.$$

After proving this, Simson adds that if BD, CD, DE are all equal, the lemma is the same as Prop. 122 of Book VII. of Pappus's *Mathematical Collection*; if CD, DE be equal, the lemma would be enunciated thus:

If from the vertex A of triangle ABC there be drawn AD to the base, then

$$AB^2 + \frac{AC^2 \cdot BD}{CD} = BD^2 + \frac{CD^2 \cdot BD}{CD} + \frac{AD^2 \cdot BC}{CD}$$

$$= BC \cdot BD \qquad + \frac{AD^2 \cdot BC}{CD}.$$

This is Stewart's theorem exactly.

Now in the appendix to the *Loci Plani* (p. 221) Simson states and proves another lemma.

Let AB be a straight line, C and D two points in it, C lying between A and B, and let CE be any straight line, then

$$\frac{AD^2 \cdot BC}{CE} + \frac{BD^2 \cdot AC}{CE} = \frac{AC^2 \cdot BC}{CE} + \frac{BC^2 \cdot AC}{CE} + \frac{CD^2 \cdot AB}{CE}$$

He then adds :†

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* Simson's mode of statement has been somewhat modernised.

† It may be well to quote Simson's own words:

Lemmate hoc usus sum . . . antequam inveneram Lem.* 10. *Alias autem ejus demonstrationes, olim, hortatu meo, excogitarunt discipuli quondam mei Dominus Jacobus Moor et Dominus Matthaeus Steuart . . . Et demonstratio quidem Dom.

"I made use of this lemma before I had discovered lemma 10. Other demonstrations of it were made out years ago by my former pupils Mr James Moor and Mr Matthew Stewart, whom I had exhorted to consider it. . . . Mr James Moor's demonstration is exactly similar to those of the 9th and 10th propositions of the second book of Euclid's *Elements*, and he has rightly remarked that these propositions are very simple cases of this lemma. Mr Matthew Stewart gave also another proof of the last case of lemma 10 in the first and second propositions of his book, *Some General Theorems, &c.*, and pointed out its use in the demonstration of some remarkable propositions."

This extract from Simson's *Loci Plani* and the extract from Stewart's preface are conclusive.

It may be added that Simson had probably discovered his theorem before 1741, because in the early part of that year he informed Matthew Stewart that he intended soon to publish his *Loci Plani.* Those who are curious to know why Simson did not carry out his intention may be referred to William Trail's *Account of the Life and Writings of Robert Simson*, pp. 24–26 (1812), and to a paper by T. S. Davies in the *Philosophical Magazine*, 3rd series, Vol. 33, pp. 201–6 (1848).

With regard to this celebrated theorem, which must now be attributed to Simson, a few historical remarks may be useful.

In 1752 Thomas Simpson proves* it in his *Select Exercises for Young Proficients in the Mathematicks.* In 1780 Euler demonstrates † it as a lemma in order to inscribe in a circle a triangle whose three sides pass through three given points.

In 1803 Carnot proves it and says ‡

"This theorem is very useful for finding the relations of the linear quantities of a figure without introducing into the calculation lineo-angular quantities. We shall often have occasion to make

---

Jac. Moor similis omnino est iis quae habentur in Prop. 9 *et Prop.* 10, *Lib.* 2 *Element. Euclidis, quas Lemmatis hujus casus esse simplicissimos recte observavit. Dom. Matth. Steuart aliam etiam ultimi casus Lem.* 10 *demonstrationem dedit in Prop.* 1 *et* 2 *libri sui de quibusdam Theorematibus generalibus, &c. . . . ejusque usum in eximiis quibusdam propositionibus demonstrandis ostendit.*

[*] See after Problem 31.

[†] *Acta Academiae . . . Petropolitanae*, Part I., pp. 92-3.

[‡] *Géométrie de Position*, pp. 262-3.

use of it, and it should be regarded as fundamental." Chasles states* that the eight lemmas of Pappus on the *Loci Plani* of Apollonius can be derived from it as corollaries or easy consequences, and he gives a generalisation of a particular case of it. Numerous and interesting applications of the theorem will be found in M. Clément Thiry's *Applications remarquables du Théorème de Stewart et Théorie du Barycentre* (1891).

The following is Simson's demonstration of the first proposition in Stewart's *General Theorems*. It is not so simple as Stewart's, but it takes account of the case when D is in the base produced. It is extracted from an unpublished letter (in my possession) of Robert Simson to Matthew Stewart. The letter is dated 13th March 1741, and is written in English, but the demonstration is in Latin.

If from the vertex A of triangle ABC there be drawn AD to the base, and DE, DF be drawn from D parallel to AC, AB; then

$$BA\cdot AE \pm CA\cdot AF = BD\cdot DC \pm AD^2$$

FIGURES 32, 33.

Draw AG making $\angle GAF$ equal to $\angle ABC$ or $\angle FDG$;
from D draw DH making $\angle ADH$ equal to $\angle AED$;
then $\angle AHD = \angle ADE = \angle DAF$.

Join GF.

Since $\angle GAF = \angle FDG$,
therefore the points G, D, A, F are concyclic.
Hence (fig. 32) $\angle FGC = \angle DAF = \angle AHD$;
therefore $\angle DGF = \angle DHB$;
and (fig. 3) $\angle FGD = \angle FAD = \angle AHD$;
therefore $\angle FGC = \angle AHD$.

Now in triangles DGF, BHD
$\angle FDG = \angle HBD$;
therefore $BD : BH = DF : DG$;
therefore $BD\cdot DG = BH\cdot DF$,
$= BH\cdot AE$.

Again because $\angle ABC = \angle GAC$,
and $\angle BDE = \angle GCA$;
therefore $BD : DE = AC : CG$;

---

* *Aperçu Historique*, 2nd ed., pp. 175-6 (1875).

therefore $BD\cdot CG = AC\cdot DE,$

$= CA\cdot AF.$

Now $BD\cdot DG = BH\cdot AE;$

therefore $BH\cdot AE + CA\cdot AF = BD\cdot DG + BD\cdot GC$

$= BD\cdot DC.$

In triangles AHD, ADE

since $\angle ADH = \angle AED$, and $\angle DAH$ is common;

therefore (both figures) $HA\cdot AE = AD^2;$

therefore $BH\cdot AE + CA\cdot AF + HA\cdot AE = BD\cdot DC + AD^2;$

therefore $BA\cdot AE + CA\cdot AF = BD\cdot DC + AD^2.$

FIGURE 33.

$BA\cdot AE + HA\cdot AE = BH\cdot AE + BD\cdot GC,$

$= BD\cdot DG + BD\cdot GC,$

$= BD\cdot DC + BD\cdot GC;$

therefore $BA\cdot AE + AD^2 = BD\cdot DC + CA\cdot AF;$

therefore $BA\cdot AE - CA\cdot AF = BD\cdot DC - AD^2.$

It may also be pointed out that the lemma which Simson employed before he had discovered Lemma 10 of the *Loci Plani*, namely,

If AB be a straight line, C and D two points in it, C lying between A *and* B, *then*

$$AD^2\cdot BC + BD^2\cdot AC = AC^2\cdot BC + BC^2.AC + CD^2\cdot AB$$

contains a theorem given by Euler in the *Novi Commentarii Academiae . . . Petropolitanæ*, vol. i., p. 49 (1747).

For $(AD^2 - AC^2)BC + (BD^2 - BC^2)AC = CD^2\cdot AB.$

But $AD - AC = BD + BC = CD;$

therefore $(AD + AC)BC + (BD - BC)AC = CD\cdot AB;$

therefore $AD\,BC + BD\cdot AC = CD\cdot AB;$

which is Euler's theorem.

Note on a property of a quadrilateral.

By Professor J. E. A. STEGGALL.

The property is that if any quadrilateral, ABCD, skew or otherwise, have its sides AB, DC divided in E, F so that

$$AE : EB = DF : FC = AD : BC,$$

then the direction of the line EF shall bisect the angle between the directions of BC, AD.

This extension of Euclid's VI. 3 follows immediately from the proposition that if AE : EB = DF : FC, then all such lines as EF are parallel to one plane, namely, the plane parallel to BC, AD; and that they each cut similar lines drawn with reference to BC, AD.

Dr Mackay has kindly supplied to me the following references bearing on the subject:—Legendre's *Geometry*, Book V., Prop. 16, (Brewster's Edition, 1824, p. 119; Hutton's *Course of Mathematics*, 12th edition, 1843, Vol. II., p. 224; *The Mathematician*, Vol. III., Supplementary Number, pp. 36–38.

Note on a possible definition of a plane.

By Professor J. E. A. STEGGALL.

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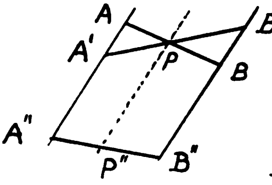

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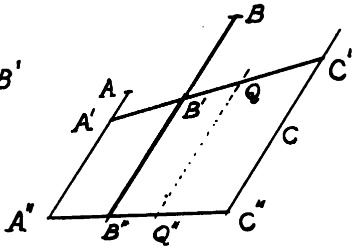

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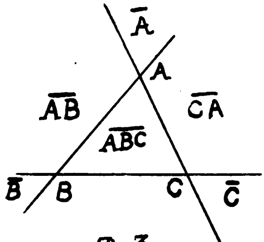

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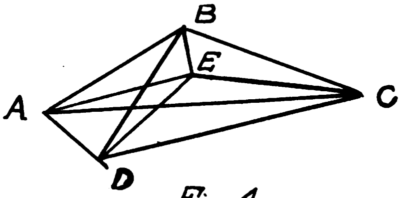

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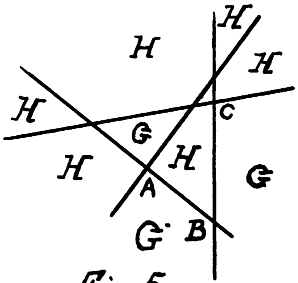

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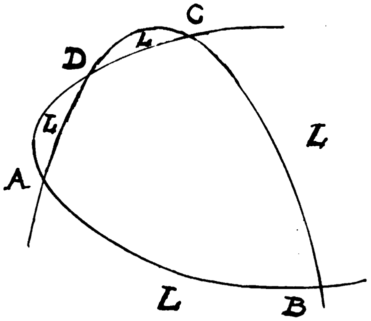

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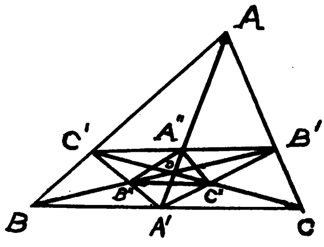

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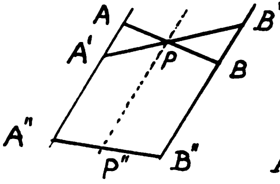

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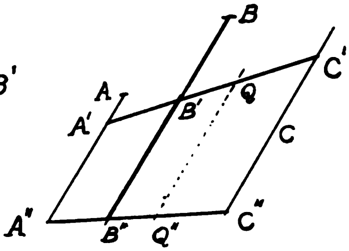

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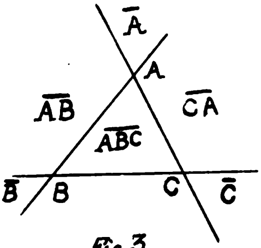

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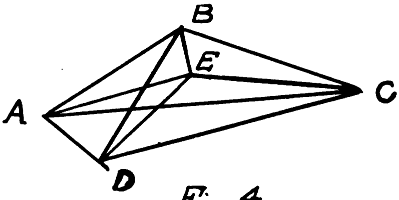

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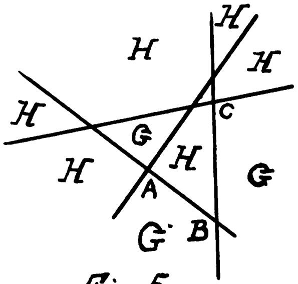

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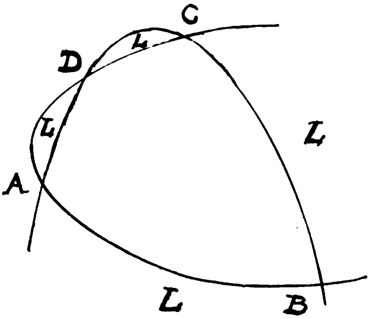

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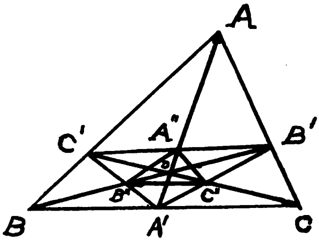

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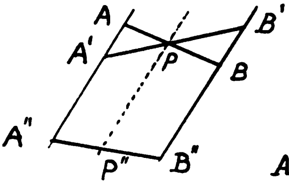

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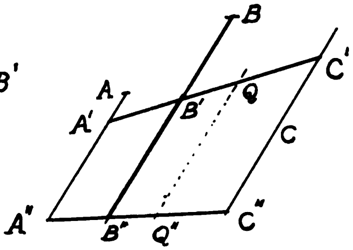

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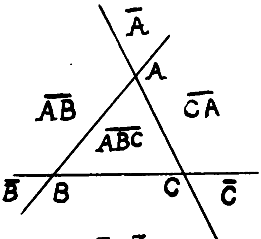

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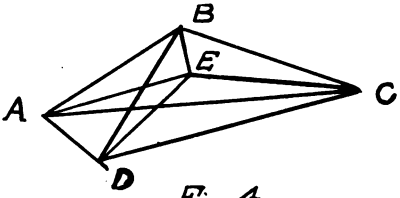

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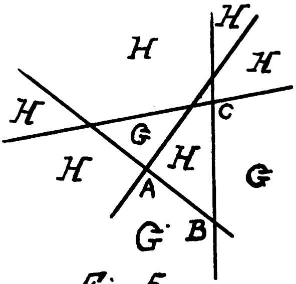

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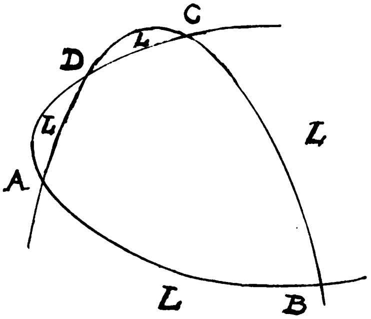

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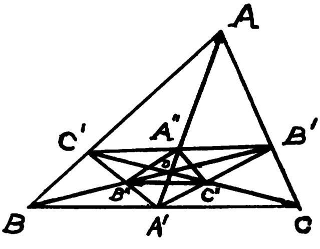
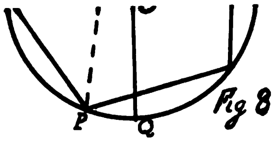


Fig. 7.



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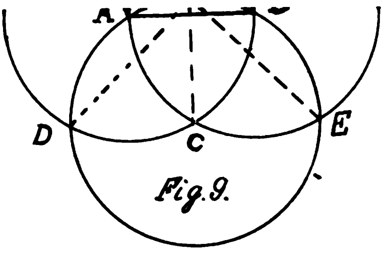

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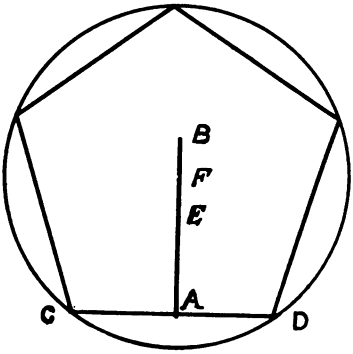

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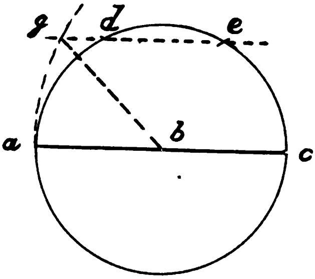

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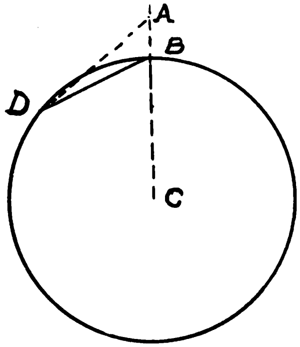

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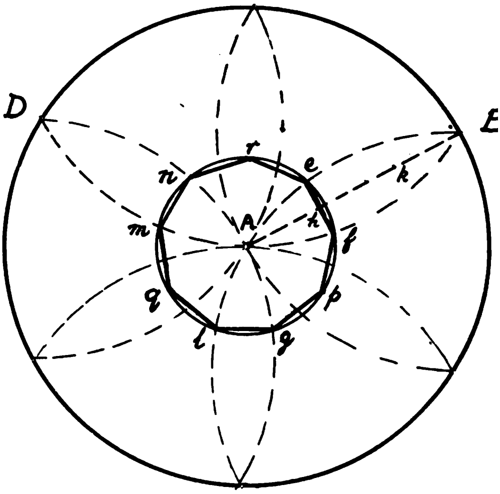

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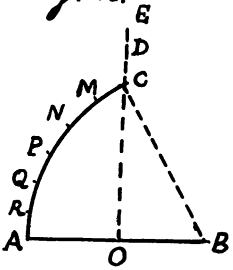

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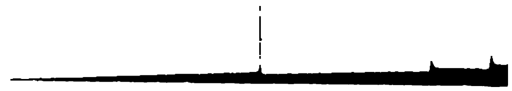

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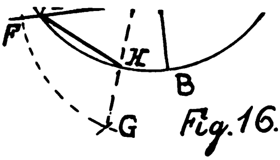

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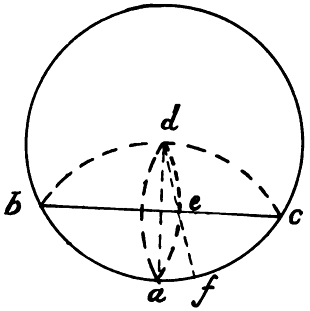
Fig 17.



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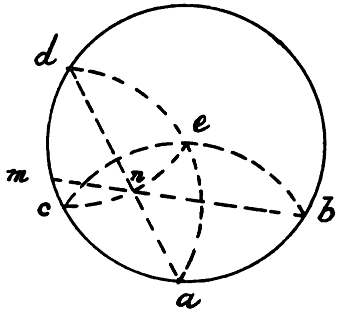

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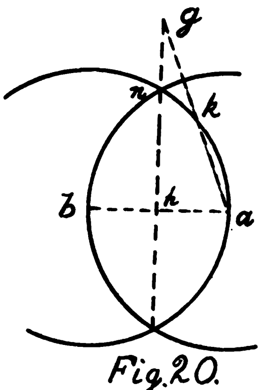

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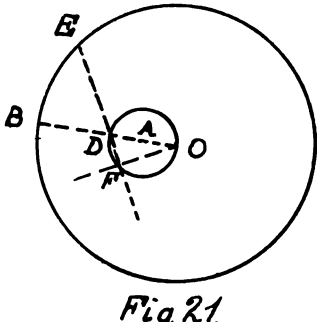

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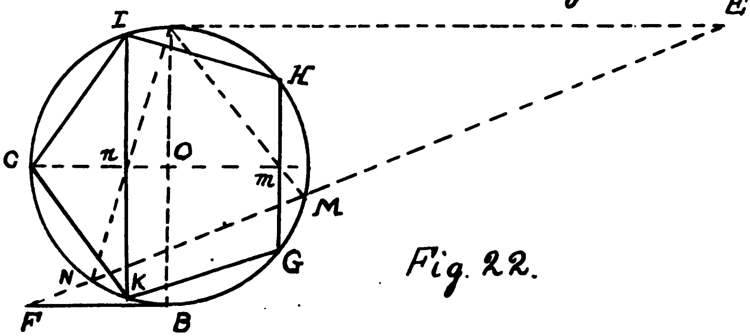

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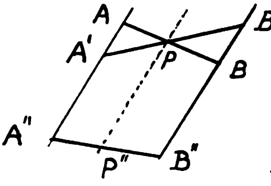

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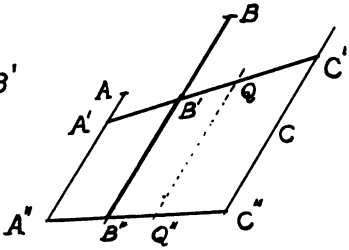

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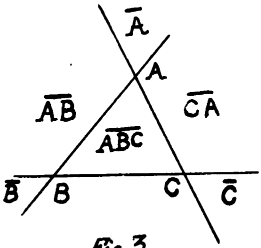

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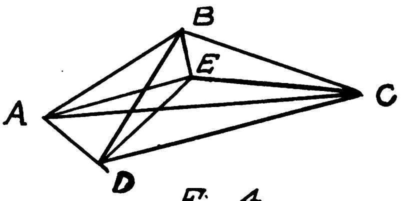

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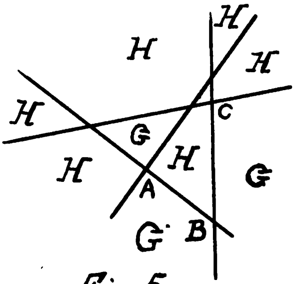

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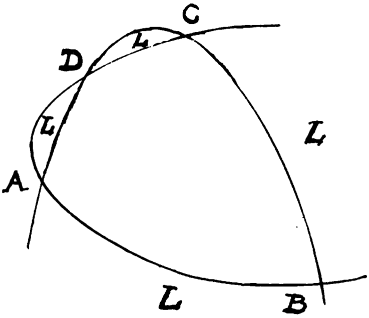

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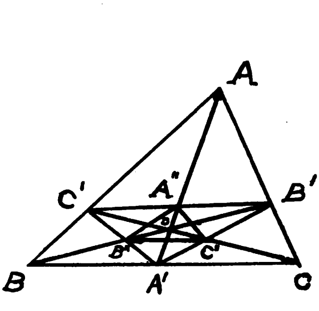

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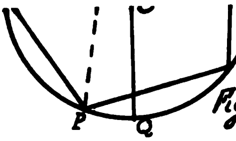

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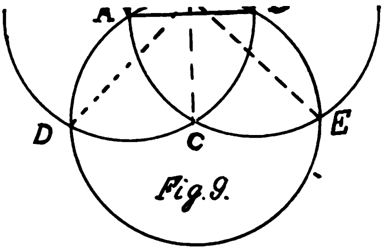

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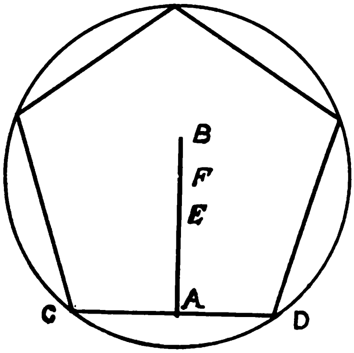

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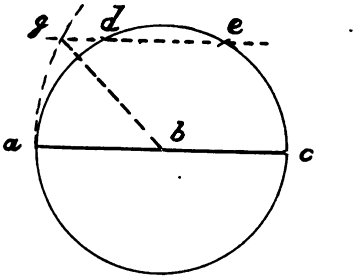

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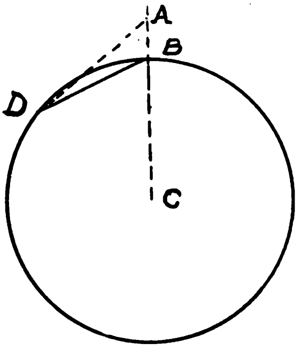

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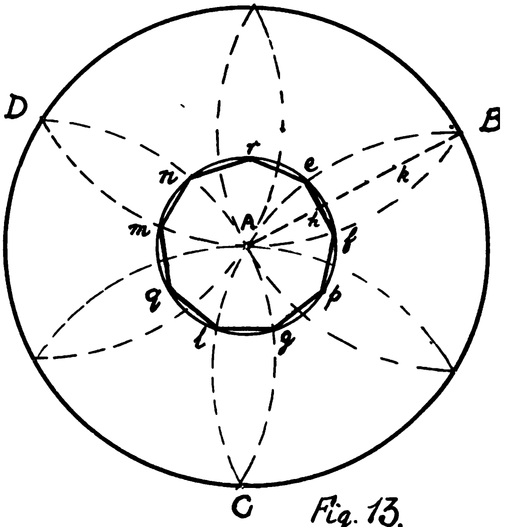

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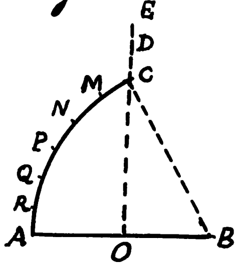

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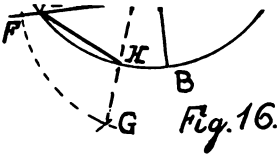
Fig 15



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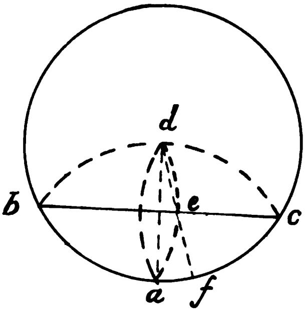

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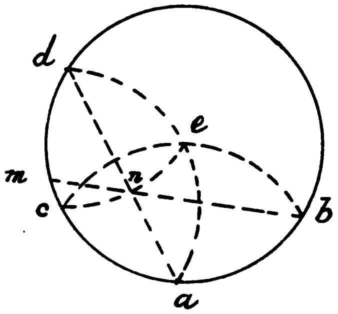

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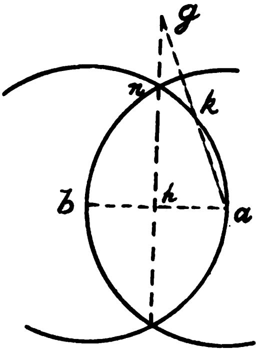

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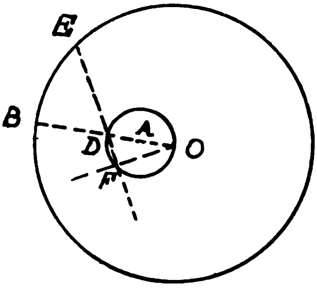

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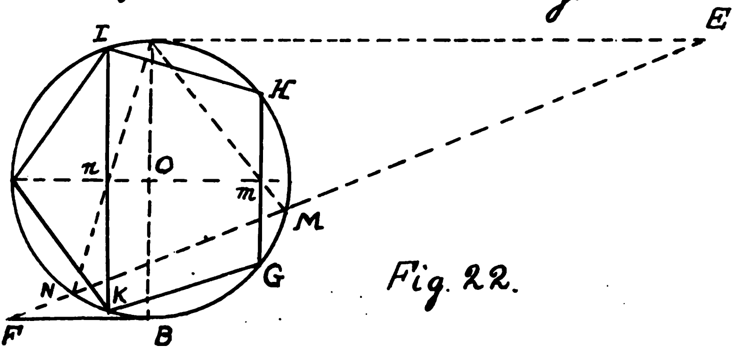

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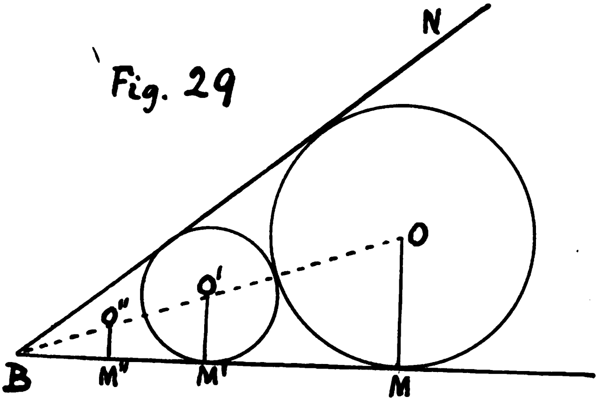

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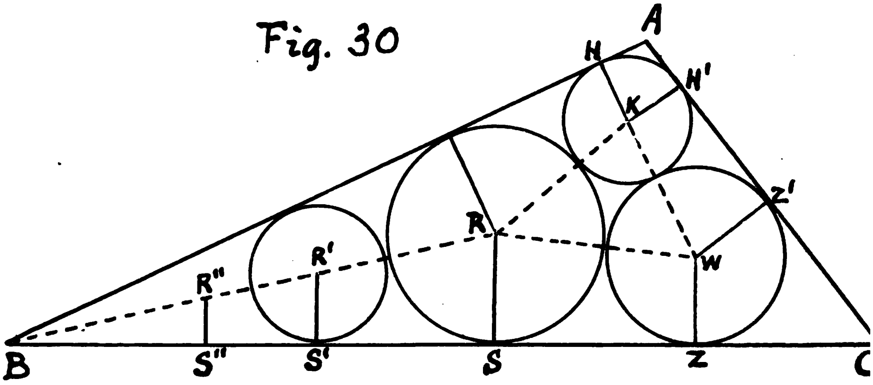

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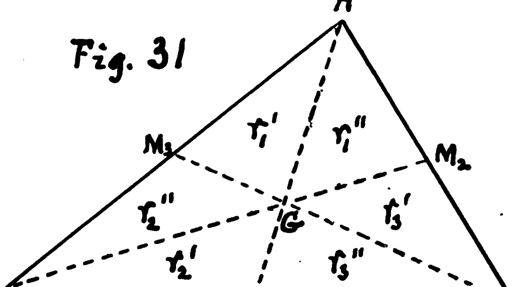

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B

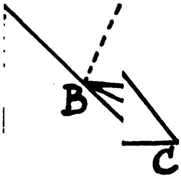
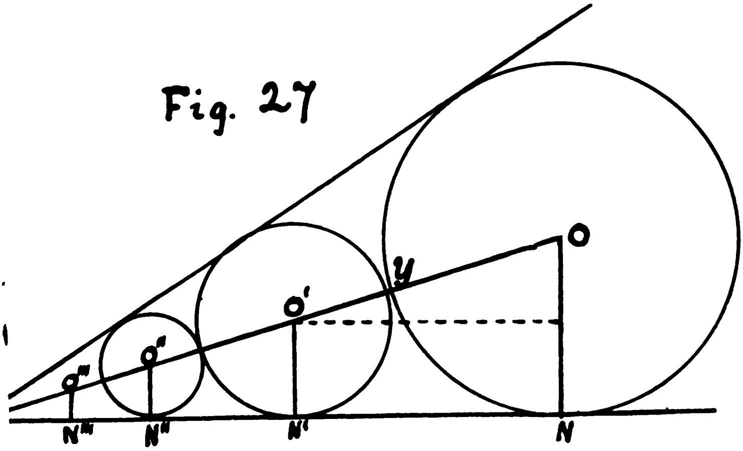

B
C

Fig. 27



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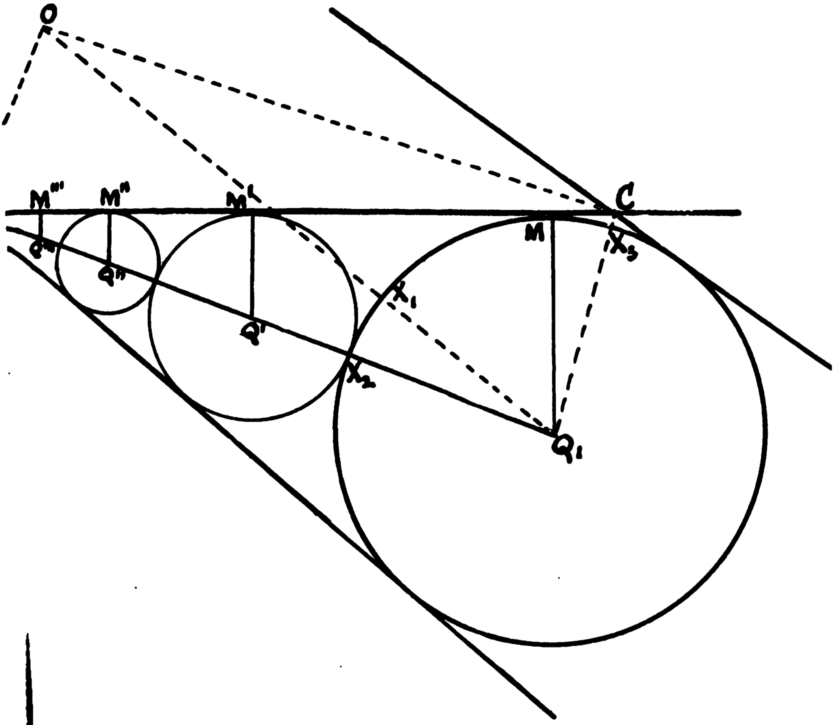

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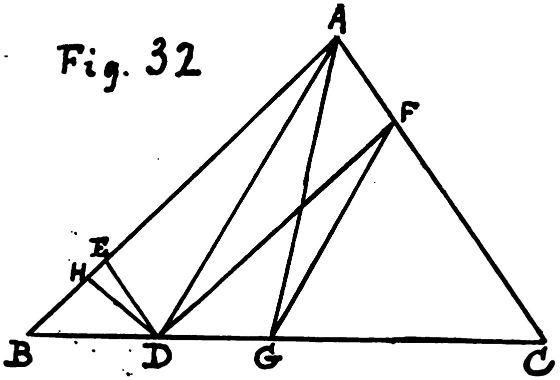

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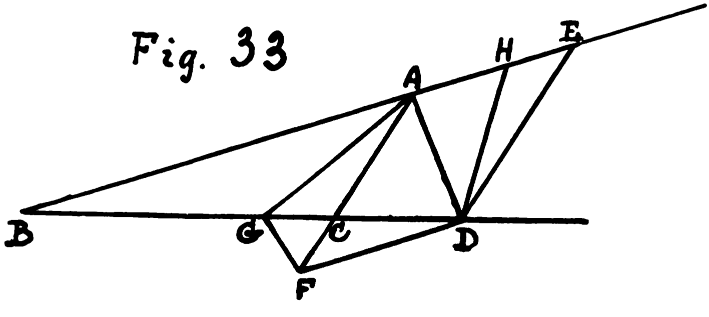

STORAGE A

STOR